Hyperspherical Functions and Harmonic Analysis on the Lorentz Group

V. V. Varlamov

Department of Mathematics, Siberia State University of Industry,

Kirova 42, Novokuznetsk 654007, Russia.

Abstract

Matrix elements of spinor and principal series representations of the Lorentz group are studied in the basis of complex angular momentum (helicity basis). It is shown that matrix elements are expressed via hyperspherical functions (relativistic spherical functions). In essence, the hyperspherical functions present itself a four-dimensional (with respect to a pseudo-euclidean metrics of Minkowski spacetime) generalization of the usual three-dimensional spherical functions. An explicit form of the hyperspherical functions is given. The hypespherical functions of the spinor representations are represented by a product of generalized spherical functions and Jacobi functions. It is shown that zonal hyperspherical functions are expressed via the Appell functions. The associated hyperspherical functions are defined as the functions on a two-dimensional complex sphere. Integral representations, addition theorems, symmetry and recurrence relations for hyperspherical functions are given. In case of the principal and supplementary series representations of the Lorentz group, the matrix elements are expressed via the functions represented by a product of spherical and conical functions. The hyperspherical functions of the principal series representations allow one to apply methods of harmonic analysis on the Lorentz group. Different forms of expansions of square integrable functions on the Lorentz group are studied. By way of example, an expansion of the wave function, representing the Dirac field $(1/2, 0)\oplus(0, 1/2)$, is given.

1 Introduction

Numerous applications of three-dimensional spherical functions in different areas of mathematical and theoretical physics are well known. In XX century, under the influence of relativity theory, which represents the world as some four-dimensional pseudo-euclidean manifold, it is appearred a necessity to generalize Laplace three-dimensional spherical functions on the four-dimensional case. Four-dimensional spherical functions of an euclidean space was first studied by Fock [21] for the solution of the hydrogen atom problem in momentum representation. They have the form

$$\psi_{nlm}(\alpha,\theta,\varphi)=\frac{i^l\sin^l\alpha}{\sqrt{n^2(n^2-1^2)\cdots(n^2-l^2)}}\frac{d^{l+1}\cos n\alpha}{d(\cos\alpha)^{l+1}}Y_{lm}(\theta,\varphi), \qquad (1)$$

where α, θ, φ are angles of the four-dimensional radius-vector in the euclidean space, $n = 0, 1, 2, \ldots$. These functions are eigenfunctions of a square of the four-dimensional angular

momentum operator L^2 (L^2 is an angular part, in the sense of α, θ, φ, of the Laplace operator).

In 1956, Dolginov [14] (see also [15, 16, 18]) considered an analytic continuation of the Fock four-dimensional spherical functions (1):

$$\psi_{Nlm}(\alpha,\theta,\varphi)=\frac{\sinh^l\alpha}{\sqrt{N^2(N^2-1^2)\cdots(N^2-l^2)}}\frac{d^{l+1}\cosh N\alpha}{d\cosh^{l+1}\alpha}Y_{lm}(\theta,\varphi), \tag{2}$$

where $0\leq\alpha\leq\infty$, $N=0,1,2,\ldots$. In turn, the functions (2), called in the works [14, 15, 16] as *relativistic spherical functions*, depend on angles of the radius-vector in the four-dimensional spacetime. At this point, if we replace in (2) N by in, where n is a real number and $0\leq n\leq\infty$, then we obtain basis functions of an irreducible unitary infinite-dimensional representation of the Lorentz group.

In 1952, Gel'fand and Z. Shapiro [22] (see also [25]) showed that matrix elements of three-dimensional rotation group are expressed via spherical functions (a relationship between special functions and group representations was first discovered by E. Cartan in 1929 [11]). The group theoretical interpretation of the theory of special functions was intensively studied by Vilenkin and Klimyk [76, 77]. At present, it is widely accepted that the theory of special functions is a 'functional aspect' of the group representation theory.

The relationship between matrix elements of the three-dimensional rotation group and spherical functions prompted a new way to definition of the four-dimensional analog of the spherical functions. Namely, these functions should be defined in terms of matrix elements of the Lorentz group. In sixties it is appearred series of the works devoted to this topics [19, 27, 61, 62, 57, 71, 38]. The works of Ström [61, 62] have been obtained the most influence. Ström used the fact, previously established by Naimark [47], that any Lorentz matrix can be represented in the form

$$\mathfrak{g}=u_1(\eta,\psi,0)\mathfrak{g}(a)u_2(\varphi_1,\theta,\varphi_2), \tag{3}$$

where $0\leq\eta<4\pi$, $0\leq\varphi_1,\varphi_2<2\pi$, $0\leq\psi,\theta\leq\pi$, $0\leq a<\infty$, and u_1, u_2 belong to the group $SU(2)$ (an universal covering of the three-dimensional rotation group), $\mathfrak{g}(a)$ is an element of the one-parameter noncompact subgroup of $SL(2,\mathbb{C})$. The decomposition (3) induces a following form for the matrix elements of the Lorentz group:

$$D^{\nu\rho}_{JM,J'M'}(\mathfrak{g})=\sum_{\lambda=-\min(J,J')}^{\min(J,J')}D^J_{M\lambda}(\eta,\psi,0)D^{\nu\rho}_{JJ'\lambda}(a)D^{J'}_{\lambda M'}(\varphi_1,\theta,\varphi_2),$$

where $D^J_{M\lambda}(\eta,\psi,0)$, $D^{J'}_{\lambda M'}(\varphi_1,\theta,\varphi_2)$ are matrix elements of $SU(2)$, and the functions $D^{\nu\rho}_{JJ'\lambda}(a)$ of the principal series, calculated by Ström in the Gel'fand-Naimark basis, have the form

$$D^{\nu\rho}_{JJ'\lambda}(a)=\sum_{d=\max(0,-\nu-\lambda)}^{\min(J-\nu,J-\lambda)}\;\sum_{d'=\max(0,-\nu-\lambda)}^{\min(J'-\nu,J'-\lambda)}B^{\nu\rho}_{JJ'\lambda dd'}e^{-a(2d'+\lambda+\nu+1-\frac{1}{2}i\rho)}\times$$
$$\times\,{}_2F_1\left(\begin{matrix}1+J'-\frac{1}{2}i\rho,\nu+\lambda+d+d'+1\\ J+J'+2\end{matrix}\middle|\,1-e^{-2a}\right),$$

where

$$B^{\nu\rho}_{JJ'\lambda dd'} = \alpha^{\nu\rho^*}_{J}\alpha^{\nu\rho}_{J'}(-1)^{J+J'-2\lambda+d+d'}\frac{1}{(J+J'+1)!}\times$$
$$[(2J+1)(J-\nu)!(J+\nu)!(J-\lambda)!(J+\lambda)!(2J'+1)(J'-\nu)!(J'+\nu)!(J'-\lambda)!(J'+\lambda)!]^{\frac{1}{2}}\times$$
$$\frac{(\nu+\lambda+d+d')!(J+J'-\nu-\lambda-d-d')!}{d!(J-\lambda-d)!(J-\nu-d)!(\lambda+\nu+d)!d'!(J'-\lambda-d')!(J'-\nu-d')!(\lambda+\nu+d')!}$$

and

$$\alpha^{\nu\rho}_{J} = \prod_{s=|\nu|}^{J}\frac{-2s+i\rho}{(4s^2+\rho^2)^{\frac{1}{2}}} = \frac{\Gamma(-|\nu|+\frac{1}{2}i\rho+1)\Gamma(-J+\frac{1}{2}i\rho)}{\Gamma(-|\nu|+\frac{1}{2}i\rho+1)\Gamma(-J+\frac{1}{2}i\rho)}.$$

Matrix elements in the form proposed by Ström and also in other forms [57, 71], which used the decomposition (3), are very complicate. An expansion problem of relativistic amplitudes requires the most simple form of the matrix elements for irreducible representations of the Lorentz group. Moreover, for the matrix elements in the Ström form or in the Sciarrino-Toller form and etc., the question about four-dimensional generalization of the Laplace spherical functions remains unclear. It should be noted that Dolginov-Toptygin relativistic spherical functions present the most degenerate form of the matrix elements of the Lorentz group. Smorodinsky and Huszar [33, 34, 60, 35, 36] found more simple method for definition (except the decomposition (3)) of the matrix elements of the Lorentz group by means of a complexification of the three-dimensional rotation group and solution of the equation on eigenvalues of the Casimir operators (see also [2]). In general case, obtained functions (later on called by Huszar [35, 36] as *spherical functions of the Lorentz group*) are products of the two hypergeometric functions. Matrix elements of spinor representations of the Lorentz group have been obtained by the author [67] via the complexification of a fundamental representation of the group $SU(2)$. The main advantage of such a definition is the most simple form of matrix elements expressed via a *hyperspherical functions*, which is a product of the generalized spherical functions P^l_{mn} and the Jacobi function $\mathfrak{P}^l_{mn}$. As is known [22], the matrix elements of $SU(2)$ are defined by the functions P^l_{mn}, and matrix elements of the group $QU(2)$ of quasiunitary matrices of the second order, which is isomorphic to the group $SL(2,\mathbb{R})$[1], are expressed via the functions $\mathfrak{P}^l_{mn}$. The hyperspherical functions present the four-dimensional analog of the three-dimensional spherical functions[2]. In essence, these functions present itself a new class of special functions related to the class of hypergeometric functions.

2 Helicity Basis

Let $\mathfrak{g}\rightarrow T_{\mathfrak{g}}$ be an arbitrary linear representation of the proper orthochronous Lorentz group $\mathfrak{G}_+$ and let $\mathsf{A}_i(t) = T_{a_i(t)}$ be an infinitesimal operator corresponding the rotation $a_i(t)\in\mathfrak{G}_+$.

[1] Other designation of this group is $SU(1,1)$ known also as three-dimensional Lorentz group, representations of which was studied by Bargmann [6].

[2] The hyperspherical functions (or hyperspherical harmonics) are known in mathematics for a long time (see, for example, [8]). These functions are generalizations of the three-dimensional spherical functions on the case of n-dimensional euclidean spaces. For that reason we retain this name (hyperspherical functions) for the case of pseudo-euclidean spaces.

Analogously, we have $\mathsf{B}_i(t)=T_{b_i(t)}$, where $b_i(t)\in\mathfrak{G}_+$ is a hyperbolic rotation. The operators A_i and B_i satisfy the following commutation relations:

$$\left.\begin{array}{lll}
[\mathsf{A}_1,\mathsf{A}_2]=\mathsf{A}_3, & [\mathsf{A}_2,\mathsf{A}_3]=\mathsf{A}_1, & [\mathsf{A}_3,\mathsf{A}_1]=\mathsf{A}_2,\\
[\mathsf{B}_1,\mathsf{B}_2]=-\mathsf{A}_3, & [\mathsf{B}_2,\mathsf{B}_3]=-\mathsf{A}_1, & [\mathsf{B}_3,\mathsf{B}_1]=-\mathsf{A}_2,\\
[\mathsf{A}_1,\mathsf{B}_1]=0, & [\mathsf{A}_2,\mathsf{B}_2]=0, & [\mathsf{A}_3,\mathsf{B}_3]=0,\\
[\mathsf{A}_1,\mathsf{B}_2]=\mathsf{B}_3, & [\mathsf{A}_1,\mathsf{B}_3]=-\mathsf{B}_2, & \\
[\mathsf{A}_2,\mathsf{B}_3]=\mathsf{B}_1, & [\mathsf{A}_2,\mathsf{B}_1]=-\mathsf{B}_3, & \\
[\mathsf{A}_3,\mathsf{B}_1]=\mathsf{B}_2, & [\mathsf{A}_3,\mathsf{B}_2]=-\mathsf{B}_1. &
\end{array}\right\}\tag{4}$$

Denoting $\mathsf{I}^{23}=\mathsf{A}_1$, $\mathsf{I}^{31}=\mathsf{A}_2$, $\mathsf{I}^{12}=\mathsf{A}_3$, and $\mathsf{I}^{01}=\mathsf{B}_1$, $\mathsf{I}^{02}=\mathsf{B}_2$, $\mathsf{I}^{03}=\mathsf{B}_3$, we can write the relations (4) in a more compact form:

$$\left[\mathsf{I}^{\mu\nu},\mathsf{I}^{\lambda\rho}\right]=\delta_{\mu\rho}\mathsf{I}^{\lambda\nu}+\delta_{\nu\lambda}\mathsf{I}^{\mu\rho}-\delta_{\nu\rho}\mathsf{I}^{\mu\lambda}-\delta_{\mu\lambda}\mathsf{I}^{\nu\rho}.$$

Let us consider the operators

$$\mathsf{X}_l=\frac{1}{2}i(\mathsf{A}_l+i\mathsf{B}_l),\quad \mathsf{Y}_l=\frac{1}{2}i(\mathsf{A}_l-i\mathsf{B}_l),\tag{5}$$
$$(l=1,2,3).$$

Using the relations (4), we find that

$$[\mathsf{X}_k,\mathsf{X}_l]=i\varepsilon_{klm}\mathsf{X}_m,\quad [\mathsf{Y}_l,\mathsf{Y}_m]=i\varepsilon_{lmn}\mathsf{Y}_n,\quad [\mathsf{X}_l,\mathsf{Y}_m]=0.\tag{6}$$

Further, introducing generators of the form

$$\left.\begin{array}{ll}
\mathsf{X}_+=\mathsf{X}_1+i\mathsf{X}_2, & \mathsf{X}_-=\mathsf{X}_1-i\mathsf{X}_2,\\
\mathsf{Y}_+=\mathsf{Y}_1+i\mathsf{Y}_2, & \mathsf{Y}_-=\mathsf{Y}_1-i\mathsf{Y}_2,
\end{array}\right\}\tag{7}$$

we see that in virtue of commutativity of the relations (6) a space of an irreducible finite–dimensional representation of the group $\mathfrak{G}_+$ can be spanned on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $\mid l,m;\dot{l},\dot{m}\rangle$, where $l,m,\dot{l},\dot{m}$ are integer or half–integer numbers, $-l\leq m\leq l$, $-\dot{l}\leq\dot{m}\leq\dot{l}$. Therefore,

$$\begin{array}{l}
\mathsf{X}_-\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(l+m)(l-m+1)}\mid l,m-1,\dot{l},\dot{m}\rangle\ \ (m>-l),\\
\mathsf{X}_+\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(l-m)(l+m+1)}\mid l,m+1;\dot{l},\dot{m}\rangle\ \ (m<l),\\
\mathsf{X}_3\mid l,m;\dot{l},\dot{m}\rangle=m\mid l,m;\dot{l},\dot{m}\rangle,\\
\mathsf{Y}_-\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}\mid l,m;\dot{l},\dot{m}-1\rangle\ \ (\dot{m}>-\dot{l}),\\
\mathsf{Y}_+\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(\dot{l}-\dot{m})(\dot{l}+\dot{m}+1)}\mid l,m;\dot{l},\dot{m}+1\rangle\ \ (\dot{m}<\dot{l}),\\
\mathsf{Y}_3\mid l,m;\dot{l},\dot{m}\rangle=\dot{m}\mid l,m;\dot{l},\dot{m}\rangle.
\end{array}\tag{8}$$

From the relations (6) it follows that each of the sets of infinitisimal operators X and Y generates the group $SU(2)$ and these two groups commute with each other. Thus, from the relations (6) and (8) it follows that the group $\mathfrak{G}_+$, in essence, is equivalent to the group $SU(2)\otimes SU(2)$. In contrast to the Gel'fand–Naimark representation for the Lorentz group [25, 47], which does not find a broad application in physics, a representation (8) is a most

useful in theoretical physics (see, for example, [1, 56, 54, 55]). This representation for the Lorentz group was first given by Van der Waerden in his brilliant book [79]. It should be noted here that the representation basis, defined by the formulae (5)–(8), has an evident physical meaning. For example, in the case of $(1,0)\oplus(0,1)$–representation space there is an analogy with the photon spin states. Namely, the operators X and Y correspond to the right and left polarization states of the photon. For that reason we will call the canonical basis consisting of the vectors $\mid lm;\dot{l}\dot{m}\rangle$ as *a helicity basis*.

As is known, a double covering of the proper orthochronous Lorentz group $\mathfrak{G}_+$, the group $SL(2,\mathbb{C})$, is isomorphic to the Clifford–Lipschitz group $\mathbf{Spin}_+(1,3)$, which, in turn, is completely defined within a biquaternion algebra $\mathbb{C}_2$, since

$$\mathbf{Spin}_+(1,3)\simeq\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathbb{C}_2:\ \ \det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=1\right\}=SL(2,\mathbb{C}).$$

Thus, a fundamental representation of the group $\mathfrak{G}_+$ is realized in a spinspace $\mathbb{S}_2$. The spinspace $\mathbb{S}_2$ is a complexification of the minimal left ideal of the algebra $\mathbb{C}_2$: $\mathbb{S}_2=\mathbb{C}\otimes I_{2,0}=\mathbb{C}\otimes C\!\ell_{2,0}e_{20}$ or $\mathbb{S}_2=\mathbb{C}\otimes I_{1,1}=\mathbb{C}\otimes C\!\ell_{1,1}e_{11}$ ($\mathbb{C}\otimes I_{0,2}=\mathbb{C}\otimes C\!\ell_{0,2}e_{02}$), where $C\!\ell_{p,q}$ ($p+q=2$) is a real subalgebra of $\mathbb{C}_2$, $I_{p,q}$ is the minimal left ideal of the algebra $C\!\ell_{p,q}$, e_{pq} is a primitive idempotent.

Linear transformations of 'vectors' (spinors and cospinors) of the spinspaces $\mathbb{S}_2$ and $\dot{\mathbb{S}}_2$ have the form

$$\begin{array}{rclcrcl} {}'\xi^1 &=& \alpha\xi^1+\beta\xi^2, & \qquad & {}'\xi^{\dot{1}} &=& \dot{\alpha}\xi^{\dot{1}}+\dot{\beta}\xi^{\dot{2}},\\ {}'\xi^2 &=& \gamma\xi^1+\delta\xi^2, & & {}'\xi^{\dot{2}} &=& \dot{\gamma}\xi^{\dot{1}}+\dot{\delta}\xi^{\dot{2}},\\ & \sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} & & & & \dot{\sigma}=\begin{pmatrix}\dot{\alpha} & \dot{\beta}\\ \dot{\gamma} & \dot{\delta}\end{pmatrix}. & \end{array} \tag{9}$$

Transformations (9) form the group $SL(2,\mathbb{C})$. The expressions (9) compose a base of the 2–spinor van der Waerden formalism [78, 53], in which the spaces $\mathbb{S}_2$ and $\dot{\mathbb{S}}_2$ are called correspondingly spaces of *undotted and dotted spinors*. The each of the spaces $\mathbb{S}_2$ and $\dot{\mathbb{S}}_2$ is homeomorphic to an extended complex plane $\mathbb{C}\cup\infty$ representing an absolute (the set of infinitely distant points) of a Lobatchevskii space $S^{1,2}$. At this point, a group of fractional linear transformations of the plane $\mathbb{C}\cup\infty$ is isomorphic to a motion group of $S^{1,2}$ [52]. Besides, in accordance with [39], the Lobatchevskii space $S^{1,2}$ is an absolute of the Minkowski world $\mathbb{R}^{1,3}$ and, therefore, the group of fractional linear transformations of the plane $\mathbb{C}\cup\infty$ (motion group of $S^{1,2}$) twice covers a 'rotation group' of the space–time $\mathbb{R}^{1,3}$, that is the proper Lorentz group.

The tensor product $\mathbb{C}_2\otimes\mathbb{C}_2\otimes\cdots\otimes\mathbb{C}_2\simeq\mathbb{C}_{2k}$ of the k algebras $\mathbb{C}_2$ induces a tensor product of the k spinspaces $\mathbb{S}_2$:

$$\mathbb{S}_2\otimes\mathbb{S}_2\otimes\cdots\otimes\mathbb{S}_2=\mathbb{S}_{2^k}.$$

Vectors of the spinspace $\mathbb{S}_{2^k}$ (or elements of the minimal left ideal of $\mathbb{C}_{2k}$) are spintensors of the following form

$$\xi^{\alpha_1\alpha_2\cdots\alpha_k}=\sum\xi^{\alpha_1}\otimes\xi^{\alpha_2}\otimes\cdots\otimes\xi^{\alpha_k}, \tag{10}$$

where summation is produced on all the index collections $(\alpha_1\ldots\alpha_k)$, $\alpha_i=1,2$. In virtue of (9) for each spinor ξ^{α_i} from (10) we have a transformation rule ${}'\xi^{\alpha_i'}=\sigma^{\alpha_i'}_{\alpha_i}\xi^{\alpha_i}$. Therefore, in general case we obtain

$${}'\xi^{\alpha_1'\alpha_2'\cdots\alpha_k'}=\sum\sigma^{\alpha_1'}_{\alpha_1}\sigma^{\alpha_2'}_{\alpha_2}\cdots\sigma^{\alpha_k'}_{\alpha_k}\xi^{\alpha_1\alpha_2\cdots\alpha_k}. \tag{11}$$

A representation (11) is called *undotted spintensor representation of the proper Lorentz group of the rank k*.

Further, let $\overset{*}{\mathbb{C}}_2$ be a biquaternion algebra, the coefficients of which are complex conjugate. Let us show that the algebra $\overset{*}{\mathbb{C}}_2$ can be obtained from $\mathbb{C}_2$ under action of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ or antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$. Indeed, in virtue of an isomorphism $\mathbb{C}_2 \simeq C\!\ell_{3,0}$ a general element

$$\mathcal{A} = a^0\mathbf{e}_0 + \sum_{i=1}^{3} a^i\mathbf{e}_i + \sum_{i=1}^{3}\sum_{j=1}^{3} a^{ij}\mathbf{e}_{ij} + a^{123}\mathbf{e}_{123}$$

of the algebra $C\!\ell_{3,0}$ can be written in the form

$$\mathcal{A} = (a^0 + \omega a^{123})\mathbf{e}_0 + (a^1 + \omega a^{23})\mathbf{e}_1 + (a^2 + \omega a^{31})\mathbf{e}_2 + (a^3 + \omega a^{12})\mathbf{e}_3, \tag{12}$$

where $\omega = \mathbf{e}_{123}$. Since ω belongs to a center of the algebra $C\!\ell_{3,0}$ (commutes with all the basis elements) and $\omega^2 = -1$, then we can to suppose $\omega \equiv i$. The action of the automorphism $\star$ on the homogeneous element $\mathcal{A}$ of a degree k is defined by a formula $\mathcal{A}^\star = (-1)^k\mathcal{A}$. In accordance with this, the action of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$, where $\mathcal{A}$ is the element (12), has a form

$$\mathcal{A} \longrightarrow \mathcal{A}^\star = -(a^0 - \omega a^{123})\mathbf{e}_0 - (a^1 - \omega a^{23})\mathbf{e}_1 - (a^2 - \omega a^{31})\mathbf{e}_2 - (a^3 - \omega a^{12})\mathbf{e}_3. \tag{13}$$

Therefore, $\star : \mathbb{C}_2 \rightarrow -\overset{*}{\mathbb{C}}_2$. Correspondingly, the action of the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ on the homogeneous element $\mathcal{A}$ of a degree k is defined by a formula $\widetilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}}\mathcal{A}$. Thus, for the element (12) we obtain

$$\mathcal{A} \longrightarrow \widetilde{\mathcal{A}} = (a^0 - \omega a^{123})\mathbf{e}_0 + (a^1 - \omega a^{23})\mathbf{e}_1 + (a^2 - \omega a^{31})\mathbf{e}_2 + (a^3 - \omega a^{12})\mathbf{e}_3, \tag{14}$$

that is, $\widetilde{\ } : \mathbb{C}_2 \rightarrow \overset{*}{\mathbb{C}}_2$. This allows us to define an algebraic analog of the Wigner's representation doubling: $\mathbb{C}_2 \oplus \overset{*}{\mathbb{C}}_2$. Further, from (12) it follows that $\mathcal{A} = \mathcal{A}_1 + \omega\mathcal{A}_2 = (a^0\mathbf{e}_0 + a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3) + \omega(a^{123}\mathbf{e}_0 + a^{23}\mathbf{e}_1 + a^{31}\mathbf{e}_2 + a^{12}\mathbf{e}_3)$. In general case, in virtue of an isomorphism $\mathbb{C}_{2k} \simeq C\!\ell_{p,q}$, where $C\!\ell_{p,q}$ is a real Clifford algebra with a division ring $\mathbb{K} \simeq \mathbb{C}$, $p-q \equiv 3, 7 \pmod 8$, we have for a general element of $C\!\ell_{p,q}$ an expression $\mathcal{A} = \mathcal{A}_1 + \omega\mathcal{A}_2$, here $\omega^2 = \mathbf{e}^2_{12\ldots p+q} = -1$ and, therefore, $\omega \equiv i$. Thus, from $\mathbb{C}_{2k}$ under action of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ we obtain a general algebraic doubling

$$\mathbb{C}_{2k} \oplus \overset{*}{\mathbb{C}}_{2k}. \tag{15}$$

Correspondingly, a tensor product $\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2 \simeq \overset{*}{\mathbb{C}}_{2r}$ of r algebras $\overset{*}{\mathbb{C}}_2$ induces a tensor product of r spinspaces $\dot{\mathbb{S}}_2$:

$$\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \cdots \otimes \dot{\mathbb{S}}_2 = \dot{\mathbb{S}}_{2^r}.$$

The vectors of the spinspace $\dot{\mathbb{S}}_{2^r}$ have the form

$$\xi^{\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r} = \sum \xi^{\dot{\alpha}_1} \otimes \xi^{\dot{\alpha}_2} \otimes \cdots \otimes \xi^{\dot{\alpha}_r}, \tag{16}$$

where the each cospinor $\xi^{\dot{\alpha}_i}$ from (16), in virtue of (9), is transformed by the rule $'\xi^{\dot{\alpha}'_i} = \sigma^{\dot{\alpha}'_i}_{\dot{\alpha}_i}\xi^{\dot{\alpha}_i}$. Therefore,

$$'\xi^{\dot{\alpha}'_1\dot{\alpha}'_2\cdots\dot{\alpha}'_r} = \sum \sigma^{\dot{\alpha}'_1}_{\dot{\alpha}_1}\sigma^{\dot{\alpha}'_2}_{\dot{\alpha}_2}\cdots\sigma^{\dot{\alpha}'_r}_{\dot{\alpha}_r}\xi^{\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r}. \tag{17}$$

A representation (17) is called *a dotted spintensor representation of the proper Lorentz group of the rank* r.

In general case we have a tensor product of k algebras $\mathbb{C}_2$ and r algebras $\overset{*}{\mathbb{C}}_2$:

$$\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2 \simeq \mathbb{C}_{2k} \otimes \overset{*}{\mathbb{C}}_{2r},$$

which induces a spinspace

$$\mathbb{S}_2 \otimes \mathbb{S}_2 \otimes \cdots \otimes \mathbb{S}_2 \otimes \dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \cdots \otimes \dot{\mathbb{S}}_2 = \mathbb{S}_{2^{k+r}} \tag{18}$$

with the vectors

$$\xi^{\alpha_1\alpha_2\cdots\alpha_k\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r} = \sum \xi^{\alpha_1} \otimes \xi^{\alpha_2} \otimes \cdots \otimes \xi^{\alpha_k} \otimes \xi^{\dot{\alpha}_1} \otimes \xi^{\dot{\alpha}_2} \otimes \cdots \otimes \xi^{\dot{\alpha}_r}. \tag{19}$$

In this case we have a natural unification of the representations (11) and (17):

$${}^\prime\xi^{\alpha_1'\alpha_2'\cdots\alpha_k'\dot{\alpha}_1'\dot{\alpha}_2'\cdots\dot{\alpha}_r'} = \sum \sigma^{\alpha_1'}_{\alpha_1}\sigma^{\alpha_2'}_{\alpha_2}\cdots\sigma^{\alpha_k'}_{\alpha_k}\sigma^{\dot{\alpha}_1'}_{\dot{\alpha}_1}\sigma^{\dot{\alpha}_2'}_{\dot{\alpha}_2}\cdots\sigma^{\dot{\alpha}_r'}_{\dot{\alpha}_r}\xi^{\alpha_1\alpha_2\cdots\alpha_k\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r}. \tag{20}$$

So, a representation (20) is called *a spintensor representation of the proper Lorentz group of the rank* (k, r).

In general case, the representations, defined by the formulas (11), (17) and (20), are reducible, that is, there exists the possibility of decomposition of the initial spinspace $\mathbb{S}_{2^{k+r}}$ (correspondingly, spinspaces $\mathbb{S}_{2^k}$ and $\mathbb{S}_{2^r}$) into a direct sum of invariant (with respect to transformations of the group $\mathfrak{G}_+$) spinspaces $\mathbb{S}_{2^{\nu_1}} \oplus \mathbb{S}_{2^{\nu_2}} \oplus \cdots \oplus \mathbb{S}_{2^{\nu_s}}$, where $\nu_1 + \nu_2 + \ldots + \nu_s = k + r$. The algebras $\mathbb{C}_2$ ($\overset{*}{\mathbb{C}}_2$) and the spinspaces $\mathbb{S}_2$ ($\dot{\mathbb{S}}_2$) correspond to fundamental represesentations $\boldsymbol{\tau}_{\frac{1}{2},0}$ ($\boldsymbol{\tau}_{0,\frac{1}{2}}$) of the Lorentz group $\mathfrak{G}_+$. In general case the spinspace (18) is reducible, that is, there exists a decomposition of the original spinspace $\mathbb{S}_{2^{k+r}}$ into a direct sum of irreducible subspaces with respect to a representation

$$\underbrace{\boldsymbol{\tau}_{\frac{1}{2},0} \otimes \boldsymbol{\tau}_{\frac{1}{2},0} \otimes \cdots \otimes \boldsymbol{\tau}_{\frac{1}{2},0}}_{k \text{ times}} \otimes \underbrace{\boldsymbol{\tau}_{0,\frac{1}{2}} \otimes \boldsymbol{\tau}_{0,\frac{1}{2}} \otimes \cdots \otimes \boldsymbol{\tau}_{0,\frac{1}{2}}}_{r \text{ times}}. \tag{21}$$

The full representation space $\mathbb{S}_{2^{k+r}}$ contains both symmetric and antisymmetric spintensors (19).

The decomposition of the spinspace with respect to $SL(2,\mathbb{C})$ is a simplest case of the Weyl scheme. Every irreducible representation of the group $SL(2,\mathbb{C})$ is defined by the Young tableau consisting of only one row. Thus, all the possible irreducible representations of $SL(2,\mathbb{C})$ correspond to the following Young tableaux:

$$\square,\ \ \square\square,\ \ \ldots,\ \ \square\square\cdots\square,\ \ \ldots$$

For that reason the representation $\boldsymbol{\tau}_{\frac{m}{2},0}$ is realized in the space $\mathrm{Sym}_{(m,0)}$ of all symmetric spintensors of the rank m. Dimension of $\mathrm{Sym}_{(m,0)}$ is equal to $m + 1$.

In turn, every element of the spinspace (18), related with the representation (21), corresponds to an element of $\mathbb{S}_{2^k} \otimes \mathbb{S}_{2^r}$ (representations $\boldsymbol{\tau}_{\frac{k}{2},0} \otimes \boldsymbol{\tau}_{\frac{r}{2},0}$ and $\boldsymbol{\tau}_{\frac{k}{2},0} \otimes \boldsymbol{\tau}_{0,\frac{r}{2}}$ are equivalent). This equivalence can be described as follows

$$\varphi \otimes \psi \longrightarrow \varphi \otimes \psi I, \tag{22}$$

where $\varphi,\psi\in\mathbb{S}_{2^k}$, $\psi I\in\mathbb{S}_{2^r}$ and

$$I=\lambda\begin{pmatrix} 0 & & -1\\ & \cdot^{\cdot^{\cdot}} & \\ (-1)_{\frac{r+k}{2}} & & 0\end{pmatrix}$$

is the matrix of a bilinear form (this matrix is symmetric if $\frac{r+k}{2}\equiv 0 \pmod 2$ and skewsymmetric if $\frac{r+k}{2}\equiv 1 \pmod 2$). In such a way, the representation (21) is realized in a symmetric space $\mathrm{Sym}_{(k,r)}$ of dimension $(k+1)(r+1)$ (or $(2l+1)(2\dot{l}+1)$ if suppose $l=k/2$ and $\dot{l}=r/2$). The decomposition of (21) is given by a Clebsh-Gordan formula

$$\boldsymbol{\tau}_{l_1\dot{l}_1}\otimes\boldsymbol{\tau}_{l_2\dot{l}_2}=\sum_{|l_1-l_2|\leq k\leq l_1+l_2;|\dot{l}_1-\dot{l}_2|\leq\dot{k}\leq\dot{l}_1+\dot{l}_2}\boldsymbol{\tau}_{k\dot{k}}.$$

where the each $\boldsymbol{\tau}_{k\dot{k}}$ acts in the space $\mathrm{Sym}_{(k,\dot{k})}$. In turn, every space $\mathrm{Sym}_{(k,r)}$ can be represented by a space of polynomials

$$p(z_0,z_1,\bar{z}_0,\bar{z}_1)=\sum_{\substack{(\alpha_1,\ldots,\alpha_k)\\(\dot{\alpha}_1,\ldots,\dot{\alpha}_r)}}\frac{1}{k!\,r!}a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}z_{\alpha_1}\cdots z_{\alpha_k}\bar{z}_{\dot{\alpha}_1}\cdots\bar{z}_{\dot{\alpha}_r}. \tag{23}$$

$$(\alpha_i,\dot{\alpha}_i=0,1)$$

where the numbers $a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}$ are unaffected at the permutations of indices. The expressions (23) can be understood as *functions on the Lorentz group*. Later on, we will find an Euler parametrization of these functions (see section 3). When the coefficients $a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}$ in (23) are depend on the variables x^α ($\alpha=0,1,2,3$), then we have

$$p(x,z,\bar{z})=\sum_{\substack{(\alpha_1,\ldots,\alpha_k)\\(\dot{\alpha}_1,\ldots,\dot{\alpha}_r)}}\frac{1}{k!\,r!}a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}(x)z_{\alpha_1}\cdots z_{\alpha_k}\bar{z}_{\dot{\alpha}_1}\cdots\bar{z}_{\dot{\alpha}_r}. \tag{24}$$

$$(\alpha_i,\dot{\alpha}_i=0,1)$$

The functions (24) should be considered as *the functions on the Poincaré group*. Some applications of these functions contained in [26]. Representations of the Poincaré group $SL(2,\mathbb{C})\odot T(4)$ are realized via the functions (24), here $T(4)$ is a subgroup of 4-dimensional translations.

Infinitesimal operators of $\mathfrak{G}_+$ in the helicity basis have a very simple form

$$\begin{aligned}
\mathsf{A}_1\mid l,m;l,\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^l_m\mid l,m-1;\dot{l},\dot{m}\rangle-\frac{i}{2}\boldsymbol{\alpha}^l_{m+1}\mid l,m+1;\dot{l}\dot{m}\rangle,\\
\mathsf{A}_2\mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^l_m\mid l,m-1;\dot{l},\dot{m}\rangle-\frac{1}{2}\boldsymbol{\alpha}^l_{m+1}\mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{A}_3\mid l,m;\dot{l},\dot{m}\rangle &= -im\mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{25}$$

$$\begin{aligned}
\mathsf{B}_1\mid l,m;\dot{l},\dot{m}\rangle &= -\frac{1}{2}\boldsymbol{\alpha}^l_m\mid l,m-1;\dot{l},\dot{m}\rangle-\frac{1}{2}\boldsymbol{\alpha}^l_{m+1}\mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{B}_2\mid l,m;\dot{l},\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^l_m\mid l,m-1;\dot{l},\dot{m}\rangle+\frac{i}{2}\boldsymbol{\alpha}^l_{m+1}\mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{B}_3\mid l,m;\dot{l},\dot{m}\rangle &= -m\mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{26}$$

$$
\begin{aligned}
\widetilde{\mathsf{A}}_1 \mid l,m;\dot{l},\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{A}}_2 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{A}}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -i\dot{m} \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\widetilde{\mathsf{B}}_1 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle + \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{B}}_2 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{B}}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -\dot{m} \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{28}
$$

where

$$
\boldsymbol{\alpha}^l_m = \sqrt{(l+m)(l-m+1)}.
$$

In the matrix notation for the operators A_i we have

$$
\mathsf{A}^j_1 = -\frac{i}{2}\begin{bmatrix}
0 & \boldsymbol{\alpha}_{-l_j+1} & 0 & \dots & 0 & 0\\
\boldsymbol{\alpha}_{-l_j+1} & 0 & \boldsymbol{\alpha}_{-l_j+2} & \dots & 0 & 0\\
0 & \boldsymbol{\alpha}_{-l_j+2} & 0 & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & \dots & 0 & \boldsymbol{\alpha}_{l_j}\\
0 & 0 & 0 & \dots & \boldsymbol{\alpha}_{l_j} & 0
\end{bmatrix}, \tag{29}
$$

$$
\mathsf{A}^j_2 = \frac{1}{2}\begin{bmatrix}
0 & \boldsymbol{\alpha}_{-l_j+1} & 0 & \dots & 0 & 0\\
-\boldsymbol{\alpha}_{-l_j+1} & 0 & \boldsymbol{\alpha}_{-l_j+2} & \dots & 0 & 0\\
0 & -\boldsymbol{\alpha}_{-l_j+2} & 0 & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & \dots & 0 & \boldsymbol{\alpha}_{l_j}\\
0 & 0 & 0 & \dots & -\boldsymbol{\alpha}_{l_j} & 0
\end{bmatrix}, \tag{30}
$$

$$
\mathsf{A}^j_3 = \begin{bmatrix}
il_j & 0 & 0 & \dots & 0 & 0\\
0 & i(l_j-1) & 0 & \dots & 0 & 0\\
0 & 0 & i(l_j-2) & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & \dots & -i(l_j-1) & 0\\
0 & 0 & 0 & \dots & 0 & -il_j
\end{bmatrix}, \tag{31}
$$

and so on.

2.1 Gel'fand-Naimark basis

There exists another representation basis for the Lorentz group:

$$
\begin{array}{rcl}
H_3\xi_{lm} & = & m\xi_{lm}, \\
H_+\xi_{lm} & = & \sqrt{(l+m+1)(l-m)}\xi_{l,m+1}, \\
H_-\xi_{lm} & = & \sqrt{(l+m)(l-m+1)}\xi_{l,m-1}, \\
F_3\xi_{lm} & = & C_l\sqrt{l^2-m^2}\xi_{l-1,m} - A_l m\xi_{l,m} - \\
 & & \qquad -C_{l+1}\sqrt{(l+1)^2-m^2}\xi_{l+1,m}, \\
F_+\xi_{lm} & = & C_l\sqrt{(l-m)(l-m-1)}\xi_{l-1,m+1} - \\
 & & \qquad -A_l\sqrt{(l-m)(l+m+1)}\xi_{l,m+1} + \\
 & & \qquad +C_{l+1}\sqrt{(l+m+1)(l+m+2)}\xi_{l+1,m+1}, \\
F_-\xi_{lm} & = & -C_l\sqrt{(l+m)(l+m-1)}\xi_{l-1,m-1} - \\
 & & \qquad -A_l\sqrt{(l+m)(l-m+1)}\xi_{l,m-1} - \\
 & & \qquad -C_{l+1}\sqrt{(l-m+1)(l-m+2)}\xi_{l+1,m-1},
\end{array}
$$

$$
A_l = \frac{il_0l_1}{l(l+1)}, \quad C_l = \frac{i}{l}\sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}},
$$

$$
m = -l, -l+1, \dots, l-1, l,
$$

$$
l = l_0\,, l_0+1, \dots,
$$

where l_0 is positive integer or half-integer number, l_1 is an arbitrary complex number. These formulas define a finite–dimensional representation of the group $\mathfrak{G}_+$ when $l_1^2 = (l_0+p)^2$, p is some natural number. In the case $l_1^2 \neq (l_0+p)^2$ we have an infinite–dimensional representation of $\mathfrak{G}_+$. The operators $H_3, H_+, H_-, F_3, F_+, F_-$ are

$$
H_+ = i\mathsf{A}_1 - \mathsf{A}_2, \quad H_- = i\mathsf{A}_1 + \mathsf{A}_2, \quad H_3 = i\mathsf{A}_3,
$$
$$
F_+ = i\mathsf{B}_1 - \mathsf{B}_2, \quad F_- = i\mathsf{B}_1 + \mathsf{B}_2, \quad F_3 = i\mathsf{B}_3.
$$

This basis was first given by Gel'fand in 1944 (see also [31, 23, 47]). The following relations between generators $\mathsf{Y}_\pm$, $\mathsf{X}_\pm$, Y_3, X_3 and $H_\pm$, $F_\pm$, H_3, F_3 define a relationship between the helicity (Van der Waerden) and Gel'fand-Naimark basises:

$$
\begin{array}{rclcrcl}
\mathsf{Y}_+ & = & -\frac{1}{2}(F_+ + iH_+), & \quad & \mathsf{X}_+ & = & \frac{1}{2}(F_+ - iH_+), \\
\mathsf{Y}_- & = & -\frac{1}{2}(F_- + iH_-), & \quad & \mathsf{X}_- & = & \frac{1}{2}(F_- - iH_-), \\
\mathsf{Y}_3 & = & -\frac{1}{2}(F_3 + iH_3), & \quad & \mathsf{X}_3 & = & \frac{1}{2}(F_3 - iH_3).
\end{array}
$$

The relation between the numbers l_0, l_1 and the number k of the factors $\mathbb{C}_2$ in the product $\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2$ is given by a following formula

$$
(l_0, l_1) = \left(\frac{k}{2}, \frac{k}{2}+1\right),
$$

whence it immediately follows that $k=l_0+l_1-1$. Thus, we have *a complex representation* $\mathfrak{C}^{l_0+l_1-1,0}$ *of the proper Lorentz group* $\mathfrak{G}_+$ *in the spinspace* $\mathbb{S}_{2^k}$. In accordance with [25] a representation conjugated to $\mathfrak{C}^{l_0+l_1-1,0}$ is defined by a pair

$$(l_0,l_1)=\left(-\frac{r}{2},\frac{r}{2}+1\right),$$

that is, this representation has a form $\mathfrak{C}^{0,l_0-l_1+1}$. In turn, a representation conjugated to fundamental representation $\mathfrak{C}^{1,0}$ is $\mathfrak{C}^{0,-1}$. As is known [25], if an irreducible representation of the proper Lorentz group $\mathfrak{G}_+$ is defined by the pair (l_0,l_1), then a conjugated representation is also irreducible and defined by a pair $\pm(l_0,-l_1)$.

2.2 One-parameter subgroups

The representation of the group $\mathfrak{G}_+$ in the space $\mathrm{Sym}(k,r)$ has a form

$$\begin{aligned} T_g q(\mathfrak{z},\bar{\mathfrak{z}}) &= \frac{1}{z_1^k\,\bar{z}_1^r}T_g\left[z_1^k\bar{z}_1^r q\left(\frac{z_0}{z_1},\frac{\bar{z}_0}{\bar{z}_1}\right)\right]= \\ &= (\gamma\mathfrak{z}+\delta)^k(\overset{*}{\gamma}\bar{\mathfrak{z}}+\overset{*}{\delta})^r q\left(\frac{\alpha\mathfrak{z}+\beta}{\gamma\mathfrak{z}+\delta},\frac{\overset{*}{\alpha}\bar{\mathfrak{z}}+\overset{*}{\beta}}{\overset{*}{\gamma}\bar{\mathfrak{z}}+\overset{*}{\delta}}\right), \end{aligned} \tag{32}$$

where

$$\mathfrak{z}=\frac{z_0}{z_1},\quad \bar{\mathfrak{z}}=\frac{\bar{z}_0}{\bar{z}_1}.$$

It is easy to see that for the group $SU(2)\subset SL(2,\mathbb{C})$ the formulae (23) and (32) reduce to the following

$$p(z_0,z_1)=\sum_{(\alpha_1,\ldots,\alpha_k)}\frac{1}{k!}a^{\alpha_1\cdots\alpha_k}z_{\alpha_1}\cdots z_{\alpha_k}, \tag{33}$$

$$T_g q(\mathfrak{z})=(\gamma\mathfrak{z}+\delta)^k q\left(\frac{\alpha\mathfrak{z}+\beta}{\gamma\mathfrak{z}+\delta}\right), \tag{34}$$

and the representation space $\mathrm{Sym}(k,r)$ reduces to $\mathrm{Sym}(k,0)$. One–parameter subgroups of $SU(2)$ are defined by the matrices

$$a_1(t)=\begin{pmatrix}\cos\frac{t}{2} & i\sin\frac{t}{2}\\ i\sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix},\quad a_2(t)=\begin{pmatrix}\cos\frac{t}{2} & -\sin\frac{t}{2}\\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix},$$

$$a_3(t)=\begin{pmatrix}e^{\frac{it}{2}} & 0\\ 0 & e^{-\frac{it}{2}}\end{pmatrix}. \tag{35}$$

An arbitrary matrix $u\in SU(2)$ written via Euler angles has a form

$$u=\begin{pmatrix}\alpha & \beta\\ -\overset{*}{\beta} & \overset{*}{\alpha}\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}e^{\frac{i(\varphi+\psi)}{2}} & i\sin\frac{\theta}{2}e^{\frac{i(\varphi-\psi)}{2}}\\ i\sin\frac{\theta}{2}e^{\frac{i(\psi-\varphi)}{2}} & \cos\frac{\theta}{2}e^{-\frac{i(\varphi+\psi)}{2}}\end{pmatrix}, \tag{36}$$

where $0 \leq \varphi < 2\pi$, $0 < \theta < \pi$, $-2\pi \leq \psi < 2\pi$, $\det u = 1$. Hence it follows that $|\alpha| = \cos\frac{\theta}{2}$, $|\beta| = \sin\frac{\theta}{2}$ and

$$\begin{aligned} \cos\theta &= 2|\alpha|^2 - 1, && (37)\\ e^{i\varphi} &= -\frac{\alpha\beta i}{|\alpha|\,|\beta|}, && (38)\\ e^{\frac{i\psi}{2}} &= \frac{\alpha e^{-\frac{i\varphi}{2}}}{|\alpha|}. && (39) \end{aligned}$$

Diagonal matrices $\begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}$ form one–parameter subgroup in the group $SU(2)$. Therefore, each matrix $u \in SU(2)$ belongs to a bilateral adjacency class containing the matrix

$$\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

The matrix element $t^l_{mn} = e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n, \psi_m\rangle$ of the group $SU(2)$ in the polynomial basis

$$\psi_n(\mathfrak{z}) = \frac{\mathfrak{z}^{l-n}}{\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)}}, \quad -l \leq n \leq l,$$

where

$$T_l(\theta)\psi(\mathfrak{z}) = \left(i\sin\frac{\theta}{2}\mathfrak{z} + \cos\frac{\theta}{2}\right)^{2l}\psi\left(\frac{\cos\frac{\theta}{2}\mathfrak{z} + i\sin\frac{\theta}{2}}{i\sin\frac{\theta}{2}\mathfrak{z} + \cos\frac{\theta}{2}}\right),$$

has a form

$$t^l_{mn}(g) = e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n, \psi_m\rangle =$$

$$\frac{e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\mathfrak{z}^{l-n}\mathfrak{z}^{l-m}\rangle}{\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}} =$$

$$e^{-i(m\varphi+n\psi)} i^{m-n}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}\times$$

$$\cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}\times$$

$$\sum_{j=\max(0,n-m)}^{\min(l-n,l+n)} \frac{i^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+n-j+1)\Gamma(m-n+j+1)}. \quad (40)$$

Further, using the formula

$${}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right| z\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k\geq 0}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\frac{z^k}{k!} \quad (41)$$

we can express the matrix element (40) via the hypergeometric function:

$$t^l_{mn}(g) = \frac{i^{m-n}e^{-i(m\varphi+n\psi)}}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\times$$

$$\cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}{}_2F_1\left(\left.\begin{matrix}m-l+1, 1-l-n\\ m-n+1\end{matrix}\right| i^2\tan^2\frac{\theta}{2}\right), \quad (42)$$

where $m \geq n$. At $m < n$ in the right part of (42) it needs to replace m and n by $-m$ and $-n$, respectively. Since l, m and n are finite numbers, then the hypergeometric series is interrupted.

Further, replacing in the one–parameter subgroups (35) the parameter t by $-it$, we obtain

$$b_1(t) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad b_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix},$$

$$b_3(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}. \tag{43}$$

These subgroups correspond to hyperbolic rotations.

3 Hyperspherical functions

As is known [28], a root subgroup of a semisimple Lie group O_4 (a rotation group of the 4-dimensional space) is a normal divisor of O_4. For that reason the 6-parameter group O_4 is semisimple, and is represented by a direct product of the two 3-parameter unimodular groups. By analogy with the group O_4, a double covering $SL(2,\mathbb{C})$ of the proper orthochronous Lorentz group $\mathfrak{G}_+$ (a rotation group of the 4-dimensional spacetime continuum) is semisimple, and is represented by a direct product of the two 3-parameter special unimodular groups, $SL(2,\mathbb{C}) \simeq SU(2)\otimes SU(2)$. An explicit form of this isomorphism can be obtained by means of a complexification of the group $SU(2)$, that is, $SL(2,\mathbb{C}) \simeq \mathsf{complex}(SU(2)) \simeq SU(2)\otimes SU(2)$ [67].

3.1 Complexification of $SU(2)$

The group $SL(2,\mathbb{C})$ of all complex matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of 2-nd order with the determinant $\alpha\delta - \gamma\beta = 1$, is a *complexification* of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2,\mathbb{C})$. The transition from $SU(2)$ to $SL(2,\mathbb{C})$ is realized via the complexification of three real parameters φ, θ, ψ (Euler angles). Let $\theta^c = \theta - i\tau$, $\varphi^c = \varphi - i\epsilon$, $\psi^c = \psi - i\varepsilon$ be complex Euler angles, where

$$\begin{array}{ccccccccccc} 0 & \leq & \operatorname{Re}\theta^c = \theta & \leq & \pi, & & -\infty & < & \operatorname{Im}\theta^c = \tau & < & +\infty, \\ 0 & \leq & \operatorname{Re}\varphi^c = \varphi & < & 2\pi, & & -\infty & < & \operatorname{Im}\varphi^c = \epsilon & < & +\infty, \\ -2\pi & \leq & \operatorname{Re}\psi^c = \psi & < & 2\pi, & & -\infty & < & \operatorname{Im}\psi^c = \varepsilon & < & +\infty. \end{array} \tag{44}$$

Replacing in (36) the angles φ, θ, ψ by the complex angles $\varphi^c, \theta^c, \psi^c$, we come to the following matrix

$$\mathfrak{g} = \begin{pmatrix} \cos\frac{\theta^c}{2} e^{\frac{i(\varphi^c+\psi^c)}{2}} & i\sin\frac{\theta^c}{2} e^{\frac{i(\varphi^c-\psi^c)}{2}} \\ i\sin\frac{\theta^c}{2} e^{\frac{i(\psi^c-\varphi^c)}{2}} & \cos\frac{\theta^c}{2} e^{-\frac{i(\varphi^c+\psi^c)}{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2} + i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right] e^{\frac{\epsilon+\varepsilon+i(\varphi+\psi)}{2}} & \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2} + i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right] e^{\frac{\epsilon-\varepsilon+i(\varphi-\psi)}{2}} \\ \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2} + i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right] e^{\frac{\varepsilon-\epsilon+i(\psi-\varphi)}{2}} & \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2} + i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right] e^{\frac{-\epsilon-\varepsilon-i(\varphi+\psi)}{2}} \end{pmatrix}, \tag{45}$$

since $\cos\frac{1}{2}(\theta-i\tau)=\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}$, and $\sin\frac{1}{2}(\theta-i\tau)=\sin\frac{\theta}{2}\cosh\frac{\tau}{2}-i\cos\frac{\theta}{2}\sinh\frac{\tau}{2}$. It is easy to verify that the matrix (45) coincides with a matrix of the fundamental reprsentation of the group $SL(2,\mathbb{C})$ (in Euler parametrization):

$$\mathfrak{g}(\varphi,\,\epsilon,\,\theta,\,\tau,\,\psi,\,\varepsilon)=$$
$$\begin{pmatrix} e^{i\frac{\varphi}{2}} & 0\\ 0 & e^{-i\frac{\varphi}{2}}\end{pmatrix}\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0\\ 0 & e^{-\frac{\epsilon}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}\\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2}\end{pmatrix}\begin{pmatrix} e^{i\frac{\psi}{2}} & 0\\ 0 & e^{-i\frac{\psi}{2}}\end{pmatrix}\begin{pmatrix} e^{\frac{\varepsilon}{2}} & 0\\ 0 & e^{-\frac{\varepsilon}{2}}\end{pmatrix}. \quad (46)$$

Moreover, the complexification of $SU(2)$ gives us the most simple and direct way for calculation of matrix elements of the Lorentz group. It is known that these elements have a great importance in quantum field theory and widely used at the study of relativistic amplitudes.

The matrix element $t^l_{mn}=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\langle T_l(\theta,\tau)\psi_\lambda,\psi_{\dot{\lambda}}\rangle$ of the finite–dimensional repsesentation of $SL(2,\mathbb{C})$ at $l=\dot{l}$ in the polynomial basis

$$\psi_\lambda(\mathfrak{z},\bar{\mathfrak{z}})=\frac{\mathfrak{z}^{l-n}\bar{\mathfrak{z}}^{l-m}}{\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-m+1)\Gamma(l+m+1)}},$$

has a form

$$t^l_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z^l_{mn}=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times$$
$$\sum^{l}_{k=-l}i^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)}\times$$
$$\cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times$$
$$\sum^{\min(l-m,l+k)}_{j=\max(0,k-m)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}\times$$
$$\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$$\sum^{\min(l-n,l+k)}_{s=\max(0,k-n)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}. \quad (47)$$

We will call the functions Z^l_{mn} in (47) as *hyperspherical functions*. Using (41), we can write the hyperspherical functions Z^l_{mn} via the hypergeometric series:

$$Z^l_{mn}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^{l}_{k=-l}i^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\left.\begin{matrix} m-l+1,1-l-k\\ m-k+1\end{matrix}\right|i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\left.\begin{matrix} n-l+1,1-l-k\\ n-k+1\end{matrix}\right|\tanh^2\frac{\tau}{2}\right). \quad (48)$$

Therefore, matrix elements can be expressed by means of the function (*a generalized hyperspherical function*)

$$\mathfrak{M}^l_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)} Z^l_{mn} e^{-n(\varepsilon+i\psi)}, \tag{49}$$

where

$$Z^l_{mn}(\cos\theta^c) = \sum_{k=-l}^{l} P^l_{mk}(\cos\theta)\mathfrak{P}^l_{kn}(\cosh\tau), \tag{50}$$

here $P^l_{mn}(\cos\theta)$ is a generalized spherical function on the group $SU(2)$ (see [25]), and $\mathfrak{P}^l_{mn}$ is an analog of the generalized spherical function for the group $QU(2)$ (so–called Jacobi function [76]). $QU(2)$ is a group of quasiunitary unimodular matrices of second order. As well as the group $SU(2)$, the group $QU(2)$ is one of the real forms of $SL(2,\mathbb{C})$ ($QU(2)$ is noncompact).

3.2 Appell functions

From (48) we see that the function Z^l_{mn} depends on two variables θ and τ. Therefore, using Bateman factorization we can express the hyperspherical functions Z^l_{mn} via Appell functions F_1–F_4 (hypergeometric series of two variables [3, 7]). Appell functions are defined by the following expressions:

$$\begin{aligned}
F_1(\alpha,\beta,\beta';\gamma;x,y) &= \sum_{m,n}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} x^m y^n,\\
F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) &= \sum_{m,n}^{\infty} \frac{(\alpha)_{m}(\beta)_m(\beta')_n}{(\gamma)_{m}(\gamma')_n m!n!} x^m y^n,\\
F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y) &= \sum_{m,n}^{\infty} \frac{(\alpha)_{m}(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} x^m y^n,\\
F_4(\alpha,\beta;\gamma,\gamma';x,y) &= \sum_{m,n}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_{m}(\gamma')_n m!n!} x^m y^n,
\end{aligned}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$. For example, for the function F_4 there exists a following factorization (see [10, 12])

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|\, x\right) {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma'\end{matrix}\middle|\, y\right) &=\\
&= \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\gamma+\gamma'-\alpha-\beta-1)_r}{r!(\gamma)_r(\gamma')_r} x^r y^r \times\\
&\qquad \times F_4(\alpha+r,\beta+r;\gamma+r,\gamma'+r;x-xy,y-xy).
\end{aligned}$$

Then, using this formula we can express the hyperspherical functions Z^l_{mn}, defining the matrix elements of the main diagonal ($m=n$), via the Appell function F_4:

$$
\begin{multlined}
Z^l_{mm}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^{l}_{k=-l}\sum_{r\geq 0}i^{m-k}(-1)^r\times\\
\frac{(m-l+1)_r(1-l-k)_r(m-k+2l-1)_r}{\Gamma(r+1)(m-k+1)^2_r}\tan^{m-k}\frac{\theta}{2}\tanh^{m-k}\frac{\tau}{2}\tan^{2r}\frac{\theta}{2}\tanh^{2r}\frac{\tau}{2}\times\\
F_4\left(m-l+r+1,1-l-k+r;m-k+r+1,m-k+r+1;-\tan^2\frac{\theta}{2}+\right.\\
\left.+\tan^2\frac{\theta}{2}\tanh^2\frac{\tau}{2},\tanh^2\frac{\tau}{2}+\tan^2\frac{\theta}{2}\tanh^2\frac{\tau}{2}\right). \quad (51)
\end{multlined}
$$

Since l and m are finite numbers, then the series standing in the right part of (51) is interrupted.

In turn, for the function F_2 there exists a following decomposition [10, 7]:

$$
\begin{multlined}
{}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right|x\right){}_2F_1\left(\left.\begin{matrix}\alpha,\beta'\\ \gamma'\end{matrix}\right|y\right)=\\
=\sum^{\infty}_{r=0}(-1)^r\frac{(\alpha)_r(\beta)_r(\beta')_r}{r!(\gamma)_r(\gamma')_r}x^ry^r\times\\
\times F_2(\alpha+r,\beta+r,\beta'+r;\gamma+r,\gamma'+r;x,y).
\end{multlined}
$$

Thus, the hyperspherical functions of the main diagonal are expressed via the Appell function F_2 as follows:

$$
\begin{multlined}
Z^l_{mm}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^{l}_{k=-l}\sum_{r\geq 0}i^{m-k}\times\\
\frac{(m-l+1)_r(1-l-k)^2_r}{\Gamma(r+1)(m-k+1)^2_r}\tan^{m-k}\frac{\theta}{2}\tanh^{m-k}\frac{\tau}{2}\tan^{2r}\frac{\theta}{2}\tanh^{2r}\frac{\tau}{2}\times\\
F_2\left(m-l+r+1,1-l-k+r,1-l-k+r;\right.\\
\left.m-k+r+1,m-k+r+1;-\tan^2\frac{\theta}{2},\tanh^2\frac{\tau}{2}\right). \quad (52)
\end{multlined}
$$

In [10], Burchnall and Chaundy obtained 15 pairs of decompositions containing the Appell functions and usual hypergeometric functions. A detailed consideration of the relationship between the hyperspherical functions Z^l_{mn} and the Appell functions F_1-F_4 (and also Horn functions [32]) comes beyond the framework of the present work and will be considered in a future paper.

3.3 Integral representations of $Z^l_{mn}(\theta,\tau)$

Let us choose in the space $\mathfrak{D}_\chi$ of the representation $T_\chi(\mathfrak{g})$ of $SU(2)\otimes SU(2)$ the basis consisting of the functions $\left\{e^{-in\vartheta}\right\}$, then

$$
T_\chi(\mathfrak{g})f(e^{i\vartheta})=(\alpha e^{i\vartheta}+\gamma)^{l+o}(\beta e^{i\vartheta}+\delta)^{l-o}f\left(\frac{\alpha e^{i\vartheta}+\gamma}{\beta e^{i\vartheta}+\delta}\right). \quad (53)
$$

Therefore,
$$T_\chi(\mathfrak{g})e^{-in\vartheta} = (\alpha e^{i\vartheta}+\gamma)^{l-n-o}(\beta e^{i\vartheta}+\delta)^{l+n+o}e^{-in\vartheta}.$$
The matrix elements $t^\chi_{mn}(\mathfrak{g})$ of the representation $T_\chi(\mathfrak{g})$ in the basis $\left\{e^{-in\vartheta}\right\}$ are Fourier coefficients in the decomposition of the functions $T_\chi(\mathfrak{g})e^{-in\vartheta}$ on the system $\left\{e^{-im\vartheta}\right\}$:

$$T_\chi(\mathfrak{g})e^{-in\vartheta} = \sum_{m=-\infty}^{\infty} t^\chi_{mn}(\mathfrak{g})e^{-im\vartheta}.$$

From the formula for the Fourier coefficients we obtain

$$t^\chi_{mn}(\mathfrak{g}) = \frac{1}{2\pi}\int\limits_0^{2\pi}(\alpha e^{i\vartheta}+\gamma)^{l-n-o}(\beta e^{i\vartheta}+\delta)^{l+n+o}e^{i(m-n)\vartheta}d\vartheta. \tag{54}$$

In such a way, we have an integral representation for the matrix elements. Making the substitution $e^{i\vartheta}=z$ we come to the following representation:

$$t^\chi_{mn}(\mathfrak{g}) = \frac{1}{2\pi i}\oint\limits_\Gamma(\alpha z+\gamma)^{l-n-o}(\beta z+\delta)^{l+n+o}z^{m-l+o-1}dz, \tag{55}$$

where $\Gamma:\ |z|=1$ is an unit circle. In the basis $\left\{e^{-im\vartheta}\right\}$ the matrices of the operators $T_\chi(h)$ have a very simple form, where
$$h=\begin{pmatrix} e^{\frac{it}{2}} & 0\\ 0 & e^{-\frac{it}{2}}\end{pmatrix}$$
is the diagonal matrix. Then from the formula (53) we see that the operator $T_\chi(h)$ has a form
$$T_\chi(h)f(e^{i\vartheta}) = e^{-i\vartheta}f(e^{i(\vartheta+t)})$$
and, therefore,
$$T_\chi(h)e^{-im\vartheta} = e^{-i(m+o)t}e^{-im\vartheta}.$$
Thus, the matrix of $T_\chi(h)$ in the basis $\left\{e^{-im\vartheta}\right\}$ is an infinite diagonal matrix, on the main diagonal of which we have the numbers $e^{-i(m+o)t}$.

As it has been shown previously (see (46)), the element $\mathfrak{g}$ of the fundamental representation of $SU(2)\otimes SU(2)$ has a form

$$\mathfrak{g}(\varphi^c,\theta^c,\psi^c) = \begin{pmatrix} e^{\frac{i\varphi^c}{2}} & 0\\ 0 & e^{-\frac{i\varphi^c}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta^c}{2} & i\sin\frac{\theta^c}{2}\\ i\sin\frac{\theta^c}{2} & \cos\frac{\theta^c}{2}\end{pmatrix}\begin{pmatrix} e^{\frac{i\psi^c}{2}} & 0\\ 0 & e^{-\frac{i\psi^c}{2}}\end{pmatrix},$$

where φ^c, θ^c, ψ^c are the complex Euler angles. Since the matrix $T_\chi(\mathfrak{g})$ for the diagonal matrices has been found, then it is remain to find a matrix of the operator $T_\chi(\mathfrak{g}(\theta,\tau))$ corresponding to the element

$$\mathfrak{g}(\theta,\tau) = \begin{pmatrix}\cos\frac{\theta^c}{2} & i\sin\frac{\theta^c}{2}\\ i\sin\frac{\theta^c}{2} & \cos\frac{\theta^c}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2} & \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\\ \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2} & \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\end{pmatrix}.$$

Then from the integral representation (54) we obtain

$$t^{\chi}_{mn}(\mathfrak{g}(\theta,\tau))=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left(\cos\frac{\theta^c}{2}e^{i\vartheta}+i\sin\frac{\theta^c}{2}\right)^{l-n-o}\left(i\sin\frac{\theta^c}{2}e^{i\vartheta}+\cos\frac{\theta^c}{2}\right)^{l+n+o}e^{i(m-n)\vartheta}d\vartheta.$$

Therefore, an integral representation for the hyperspherical function $Z^l_{mn}(\theta,\tau)$ has a form

$$Z^l_{mn}(\theta,\tau)=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left(\cos\frac{\theta^c}{2}e^{i\vartheta}+i\sin\frac{\theta^c}{2}\right)^{l-n}\left(i\sin\frac{\theta^c}{2}e^{i\vartheta}+\cos\frac{\theta^c}{2}\right)^{l+n}e^{i(m-n)\vartheta}d\vartheta=$$
$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left(\left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]e^{i\vartheta}+\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right)^{l-n}\times$$
$$\left(\left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]e^{i\vartheta}+\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right)^{l+n}e^{i(m-n)\vartheta}d\vartheta. \quad (56)$$

Or, using the integral representation (55), we obtain

$$Z^l_{mn}(\theta,\tau)=\frac{1}{2\pi i}\oint\limits_{\Gamma}\left(\cos\frac{\theta^c}{2}z+i\sin\frac{\theta^c}{2}\right)^{l-n}\left(i\sin\frac{\theta^c}{2}z+\cos\frac{\theta^c}{2}\right)^{l+n}z^{m-l-1}dz=$$
$$\frac{1}{2\pi i}\oint\limits_{\Gamma}\left(\left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]z+\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right)^{l-n}\times$$
$$\left(\left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]z+\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right)^{l+n}z^{m-l-1}dz. \quad (57)$$

3.4 Zonal hyperspherical functions

Let $o=0$, then in the space $\mathfrak{D}_l$ there is a function which invariant with respect to all the operators $T_l(h)$, where h is the diagonal matrix from the group $SL(2,\mathbb{C})\sim SU(2)\otimes SU(2)$:

$$h=\begin{pmatrix} e^{\frac{it}{2}} & 0\\ 0 & e^{-\frac{it}{2}}\end{pmatrix}.$$

For example, the basis element 1 is such a function. The matrix element $t^l_{00}(\mathfrak{g})$, corresponding to such a basis element, we will call *a zonal hyperspherical function* of the representation $T_l(\mathfrak{g})$ with respect to the subgroup Ω of the diagonal matrices.

So, at $o=0$ the representation $T_l(\mathfrak{g})$ has a zonal hyperspherical function satisfying the following equation

$$t^l_{00}(h_1\mathfrak{g}h_2)=t^l_{00}(\mathfrak{g}),$$

where $h_1,h_2\in\Omega$. Hence it immediately follows that $t^l_{00}(\mathfrak{g})$ depends only on the Euler angles θ and τ. From (47) we obtain

$$t^l_{00}(\mathfrak{g})=Z^l_{00}(\theta,\tau).$$

We will denote the function $Z^l_{00}(\theta,\tau)$ via $Z_l(\theta,\tau)$. Thus,

$$Z_l(\theta,\tau)=t^l_{00}(0,0,\theta,\tau,0,0)=Z^l_{00}(\theta,\tau).$$

From (47) it follows an explicit expression for the zonal hypersperical function:

$$
Z_l(\theta,\tau)=\sum_{k=-l}^{l} i^{-k}\Gamma(l+1)\sqrt{\Gamma(l-k+1)\Gamma(l+k+1)}\times
$$

$$
\cos^{2l}\frac{\theta}{2}\tan^{-k}\frac{\theta}{2}\times
$$

$$
\sum_{j=\max(0,k)}^{\min(l,l+k)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-j+1)\Gamma(l+k-j+1)\Gamma(j-k+1)}\times
$$

$$
\Gamma(l+1)\sqrt{\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{-k}\frac{\tau}{2}\times
$$

$$
\sum_{s=\max(0,k)}^{\min(l,l+k)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(l-s+1)\Gamma(l+k-s+1)\Gamma(s-k+1)}.
$$

Further, from (48), (51) and (52) we obtain corresponding expressions for $Z_l(\theta,\tau)$ via the hypergeometric function ${}_2F_1$ and the Appell functions F_4 and F_2:

$$
Z^l_{mn}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{-k}\tan^{-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\times
$$

$$
{}_2F_1\left(\begin{matrix}-l+1,1-l-k\\ -k+1\end{matrix}\,\middle|\,i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-l+1,1-l-k\\ -k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),
$$

$$
Z^l_{mm}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\sum_{r\geq 0} i^{-k}(-1)^r\times
$$

$$
\frac{(-l+1)_r(1-l-k)_r(2l-k-1)_r}{\Gamma(r+1)(-k+1)^2_r}\tan^{-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\tan^{2r}\frac{\theta}{2}\tanh^{2r}\frac{\tau}{2}\times
$$

$$
F_4\left(r-l+1,1-l-k+r;r-k+1,r-k+1;-\tan^2\frac{\theta}{2}+\right.
$$

$$
\left.+\tan^2\frac{\theta}{2}\tanh^2\frac{\tau}{2},\tanh^2\frac{\tau}{2}+\tan^2\frac{\theta}{2}\tanh^2\frac{\tau}{2}\right),
$$

$$
Z^l_{mm}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\sum_{r\geq 0} i^{-k}\times
$$

$$
\frac{(-l+1)_r(1-l-k)^2_r}{\Gamma(r+1)(-k+1)^2_r}\tan^{-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\tan^{2r}\frac{\theta}{2}\tanh^{2r}\frac{\tau}{2}\times
$$

$$
F_2\left(r-l+1,1-l-k+r,1-l-k+r;r-k+1,r-k+1;-\tan^2\frac{\theta}{2},\tanh^2\frac{\tau}{2}\right).
$$

Integral representations (56) and (57) have in this case the following form

$$Z_l(\theta,\tau)=\frac{1}{2\pi}\int\limits_0^{2\pi}\left(\cos\frac{\theta^c}{2}e^{i\vartheta}+i\sin\frac{\theta^c}{2}\right)^l\left(i\sin\frac{\theta^c}{2}e^{i\vartheta}+\cos\frac{\theta^c}{2}\right)^l d\vartheta=$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}(\cos\theta^c+i\sin\theta^c\cos\vartheta)d\vartheta=$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}(\cos\theta\cosh\tau+i\sin\theta\sinh\tau+i[\sin\theta\cosh\tau-i\cos\theta\sinh\tau]\cos\vartheta)d\vartheta,$$

$$Z_l(\theta,\tau)=\frac{1}{2\pi i}\oint\limits_{\Gamma}\left(\cos\frac{\theta^c}{2}z+i\sin\frac{\theta^c}{2}\right)^l\left(i\sin\frac{\theta^c}{2}z+\cos\frac{\theta^c}{2}\right)^l z^{-l-1}dz=$$

$$\frac{i^l\sin^l\theta^c}{2^{l+1}\pi i}\oint\limits_{\Gamma}(z^2-2iz\cot\theta^c+1)^l z^{-l-1}dz=$$

$$\frac{i^l(\sin\theta\cosh\tau-i\cos\theta\sinh\tau)^l}{2^{l+1}\pi i}\oint\limits_{\Gamma}(z^2-2iz\frac{\cot\theta\coth\tau+1}{\coth\tau-\cot\theta}+1)z^{-l-1}dz.$$

3.5 Associated hyperspherical functions

Let us consider now associated hyperspherical functions of the representation $T_\chi(\mathfrak{g})$, $\chi=(l,0)$, that is, the matrix elements $t^l_{m0}(\mathfrak{g})$ standing in one column with the function $t^l_{00}(\mathfrak{g})$. In this case from (47) we have

$$t^l_{m0}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}Z^l_{m0}(\theta,\tau).$$

Hence it follows that matrix elements $t^l_{m0}(\mathfrak{g})$ do not depend on the Euler angles ε and ψ, that is, $t^l_{m0}(\mathfrak{g})$ are constant on the left adjacency classes formed by the subgroup Ω^c_ψ of the diagonal matrices $\begin{pmatrix} e^{\frac{i\psi^c}{2}} & 0\\ 0 & e^{-\frac{i\psi^c}{2}}\end{pmatrix}$. Therefore,

$$t^l_{m0}(\mathfrak{g}h)=t^l_{m0}(\mathfrak{g}),\quad h\in\Omega^c_\psi.$$

We will denote the functions $Z^{m0}(\theta,\tau)$ via $Z_l^m(\theta,\tau)$. From (47) we obtain an explicit expression for the *associated hyperspherical function* $Z_l^m(\theta,\tau)$:

$$Z_l^m(\theta,\tau)=\sum_{k=-l}^{l} i^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)}\times$$
$$\cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times$$
$$\sum_{j=\max(0,k-m)}^{\min(l-m,l+k)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}\times$$
$$\Gamma(l+1)\sqrt{\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{-k}\frac{\tau}{2}\times$$
$$\sum_{s=\max(0,k)}^{\min(l,l+k)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(l-s+1)\Gamma(l+k-s+1)\Gamma(s-k+1)}. \qquad (58)$$

Further, from (48) it follows that

$$Z^l_{mn}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{matrix}m-l+1,1-l-k\\ m-k+1\end{matrix}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-l+1,1-l-k\\ -k+1\end{matrix}\middle|\,\tanh^2\frac{\tau}{2}\right). \qquad (59)$$

Corresponding integral representations for the functions $Z_l^m(\theta,\tau)$ have the form:

$$Z_l^m(\theta,\tau)=\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta^c+i\sin\theta^c\cos\vartheta)e^{im\vartheta}d\vartheta=$$
$$\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta\cosh\tau+i\sin\theta\sinh\tau+i[\sin\theta\cosh\tau-i\cos\theta\sinh\tau]\cos\vartheta)e^{im\vartheta}d\vartheta,$$

$$Z_l^m(\theta,\tau)=\frac{i^l\sin^l\theta^c}{2^{l+1}\pi i}\oint_\Gamma(z^2-2iz\cot\theta^c+1)^l z^{m-l-1}dz=$$
$$\frac{i^l(\sin\theta\cosh\tau-i\cos\theta\sinh\tau)^l}{2^{l+1}\pi i}\oint_\Gamma(z^2-2iz\frac{\cot\theta\coth\tau+1}{\coth\tau-\cot\theta}+1)z^{m-l-1}dz.$$

Associated hyperspherical functions admit a very elegant geometric interpretation, namely, they are the functions on the surface of a two-dimensional complex sphere. Indeed, let us construct in $\mathbb{C}^3$ a two–dimensional complex sphere from the quantities $z_k=x_k+iy_k$, $\overset{*}{z}_k=x_k-iy_k$ as follows (see Figure 1)

$$\mathbf{z}^2=z_1^2+z_2^2+z_3^2=\mathbf{x}^2-\mathbf{y}^2+2i\mathbf{x}\mathbf{y}=r^2 \qquad (60)$$

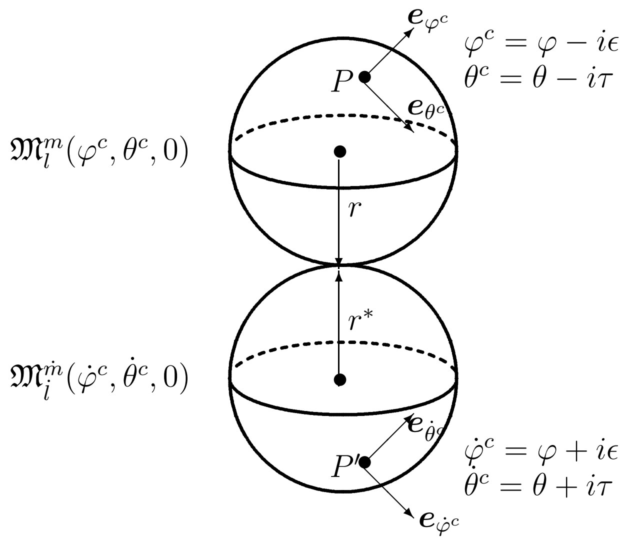

Figure 1 Two–dimensional complex sphere $z_1^2 + z_2^2 + z_3^2 = r^2$ in three–dimensional complex space $\mathbb{C}^3$. The space $\mathbb{C}^3$ is isometric to the bivector space $\mathbb{R}^6$. The dual (complex conjugate) sphere $\overset{*}{z}_1{}^2 + \overset{*}{z}_2{}^2 + \overset{*}{z}_3{}^2 = \overset{*}{r}{}^2$ is a mirror image of the complex sphere with respect to the hyperplane. The associated hyperspherical functions $\mathfrak{M}_l^m(\varphi^c, \theta^c, 0)$ ($\mathfrak{M}_{\dot{l}}^{\dot{m}}(\dot{\varphi}^c, \dot{\theta}^c, 0)$) are defined on the surface of the complex (dual) sphere.

and its complex conjugate (dual) sphere

$$\overset{*}{\mathbf{z}}{}^2 = \overset{*}{z}_1{}^2 + \overset{*}{z}_2{}^2 + \overset{*}{z}_3{}^2 = \mathbf{x}^2 - \mathbf{y}^2 - 2i\mathbf{x}\mathbf{y} = \overset{*}{r}{}^2. \tag{61}$$

For more details about the two-dimensional complex sphere see [33, 34, 60]. It is well-known that both quantities $\mathbf{x}^2 - \mathbf{y}^2$, $\mathbf{x}\mathbf{y}$ are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant (Casimir operators of the Lorentz group are constructed from such quantities, see also (101)). It is easy to see that three–dimensional complex space $\mathbb{C}^3$ is isometric to a real space $\mathbb{R}^{3,3}$ with a basis $\{i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$. At this point a metric tensor of $\mathbb{R}^{3,3}$ has the form

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence it immediately follows that $\mathbb{C}^3$ is isometric to the bivector space $\mathbb{R}^6$.

3.6 Symmetry relations for the functions $Z^l_{mn}(\theta, \tau)$

The hyperspherical functions $Z^l_{mn}(\theta, \tau)$ satisfy some symmetry relations with respect to the indices l, m, n. Let us show that these functions satisfy the relation

$$Z^l_{mn}(\theta, \tau) = Z^l_{-m,-n}(\theta, \tau). \tag{62}$$

Consider the operator S transforming $f(e^{i\vartheta})$ into $e^{2io\vartheta}f(e^{-i\vartheta})$. It is easy to verify that S commutes with an operator $T_\chi(\mathfrak{g}(\theta,\tau))$, where

$$\mathfrak{g}(\theta,\tau)=\begin{pmatrix}\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2} & \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\\ \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2} & \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\end{pmatrix},$$

that is,

$$ST_\chi(\mathfrak{g}(\theta,\tau))=T_\chi(\mathfrak{g}(\theta,\tau))S. \tag{63}$$

Matrix elements s_{mn} of the operator S in the basis $\{e^{-im\vartheta}\}$ equal to zero if $m+n\neq -2o$, and equal to the unit if $m+n=-2o$. Multiplying the matrices in (63) and comparing the corresponding elements, we obtain

$$t^\chi_{-2o-m,n}(\mathfrak{g}(\theta,\tau))=t^\chi_{m,-2o-n}(\mathfrak{g}(\theta,\tau)).$$

Since $t^\chi_{mn}(\mathfrak{g}(\theta,\tau))=Z^l_{m',n'}(\theta,\tau)$, where $m'=m+o$, $n'=n+o$, then

$$Z^l_{-m-o,n+o}(\theta,\tau)=Z^l_{m+o,-n-o}(\theta,\tau).$$

Hence it follows that

$$Z^l_{mn}(\theta,\tau)=Z^l_{-m,-n}(\theta,\tau)$$

for all the complex values of l and all integer or half-integer values of m and n. Further, replacing in the expression (47) m by $-n$, and n by $-m$, we obtain

$$Z^l_{-n,-m}(\theta,\tau)=Z^l_{m,n}(\theta,\tau)$$

or, in virtue of the relation (62),

$$Z^l_{nm}(\theta,\tau)=Z^l_{mn}(\theta,\tau). \tag{64}$$

Therefore, the hyperspherical functions $Z^l_{mn}(\theta,\tau)$ are symmetric.

3.7 Matrices $T_l(\mathfrak{g})$

Using the formula (47), let us find explicit expressions for the matrices $T_l(\mathfrak{g})$ of the finite–dimensional representations of $\mathfrak{G}_+$ at $l=0,\frac{1}{2},1$:

$$T_0(\theta,\tau)=1,$$

$$T_{\frac{1}{2}}(\theta,\tau)=\begin{pmatrix}Z^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} & Z^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}\\ Z^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & Z^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}\end{pmatrix}=$$

$$\begin{pmatrix}\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2} & \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\\ \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2} & \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\end{pmatrix}, \tag{65}$$

$$T_1(\theta,\tau)=\begin{pmatrix}Z^1_{-1-1} & Z^1_{-10} & Z^1_{-11}\\ Z^1_{0-1} & Z^1_{00} & Z^1_{01}\\ Z^1_{1-1} & Z^1_{10} & Z^1_{11}\end{pmatrix}=$$

$$\begin{pmatrix} \cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2} & \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) & \cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2} \\ \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) & \cos\theta\cosh\tau+i\sin\theta\sinh\tau & \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) \\ \cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2} & \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) & \cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2} \end{pmatrix}. \quad (66)$$

3.8 Addition Theorem

Let $\mathfrak{g}=\mathfrak{g}_1\mathfrak{g}_2$ be the product of two matrices $\mathfrak{g}_1$, $\mathfrak{g}_2\in SL(2,\mathbb{C})$. Let us denote the Euler angles of the matrix $\mathfrak{g}$ via $\varphi^c,\theta^c,\psi^c$, of the matrix $\mathfrak{g}_1$ via $\varphi^c_1,\theta^c_1,\psi^c_1$ and of the matrix $\mathfrak{g}_2$ via $\varphi^c_2,\theta^c_2,\psi^c_2$. Expressing now the Euler angles of the matrix $\mathfrak{g}$ via the Euler angles of the factors $\mathfrak{g}_1$, $\mathfrak{g}_2$, we consider at first the particular case $\varphi^c_1=\psi^c_1=\psi^c_2=0$:

$$\mathfrak{g}=\begin{pmatrix}\cos\frac{\theta^c_1}{2} & i\sin\frac{\theta^c_1}{2}\\ i\sin\frac{\theta^c_1}{2} & \cos\frac{\theta^c_1}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta^c_2}{2}e^{\frac{i\varphi^c_2}{2}} & i\sin\frac{\theta^c_2}{2}e^{\frac{i\varphi^c_2}{2}}\\ i\sin\frac{\theta^c_2}{2}e^{-\frac{i\varphi^c_2}{2}} & \cos\frac{\theta^c_2}{2}e^{-\frac{i\varphi^c_2}{2}}\end{pmatrix}.$$

Multiplying the matrices in the right part of this equality and using a complex analog of the formulae (37)–(39), we obtain

$$\cos\theta^c = \cos\theta^c_1\cos\theta^c_2-\sin\theta^c_1\sin\theta^c_2\cos\varphi^c_2, \quad (67)$$

$$e^{i\varphi^c} = \frac{\sin\theta^c_1\cos\theta^c_2+\cos\theta^c_1\sin\theta^c_2\cos\varphi^c_2+i\sin\theta^c_2\sin\varphi^c_2}{\sin\theta^c}, \quad (68)$$

$$e^{\frac{i(\varphi^c+\psi^c)}{2}} = \frac{\cos\frac{\theta^c_1}{2}\cos\frac{\theta^c_2}{2}e^{\frac{i\varphi^c_2}{2}}-\sin\frac{\theta^c_1}{2}\sin\frac{\theta^c_2}{2}e^{-\frac{i\varphi^c_2}{2}}}{\cos\frac{\theta^c}{2}}. \quad (69)$$

It is not difficult to obtain a general case. Indeed, in virtue of (46) the matrix $\mathfrak{g}\in SL(2,\mathbb{C})$ admits a representation

$$\begin{aligned}\mathfrak{g}(\varphi^c,\theta^c,\psi^c) &= \begin{pmatrix}e^{\frac{i\varphi^c}{2}} & 0\\ 0 & e^{-\frac{i\varphi^c}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta^c}{2} & i\sin\frac{\theta^c}{2}\\ i\sin\frac{\theta^c}{2} & \cos\frac{\theta^c}{2}\end{pmatrix}\begin{pmatrix}e^{\frac{i\psi^c}{2}} & 0\\ 0 & e^{-\frac{i\psi^c}{2}}\end{pmatrix}\equiv\\ &\equiv \mathfrak{g}(\varphi^c,0,0)\mathfrak{g}(0,\theta^c,0)\mathfrak{g}(0,0,\psi^c).\end{aligned}$$

Therefore,

$$\mathfrak{g}(\varphi^c_1,\theta^c_1,\psi^c_1)\mathfrak{g}(\varphi^c_2,\theta^c_2,\psi^c_2)= \mathfrak{g}(\varphi^c_1,0,0)\mathfrak{g}(0,\theta^c_1,0)\mathfrak{g}(0,0,\psi^c_1)\mathfrak{g}(\varphi^c_2,0,0)\mathfrak{g}(0,\theta^c_2,0)\mathfrak{g}(0,0,\psi^c_2). \quad (70)$$

It is obvious that

$$\mathfrak{g}(0,0,\psi^c_1)\mathfrak{g}(\varphi^c_2,0,0)=\mathfrak{g}(\varphi^c_2+\psi^c_1,0,0).$$

Besides, if we multiply the matrix $\mathfrak{g}(\varphi^c,\theta^c,\psi^c)$ at the left by the matrix $\mathfrak{g}(\varphi^c_1,0,0)$, the Euler angle φ^c increases by φ^c_1, and other Euler angles remain unaltered. Analogously, if we multiply at the right the matrix $\mathfrak{g}(\varphi^c,\theta^c,\psi^c)$ by $\mathfrak{g}(0,0,\psi^c_2)$, the angle ψ^c increases by ψ^c_2. Hence it follows that in general case the angle φ^c_2 should be replaced by $\varphi^c_2+\psi^c_1$, and the angles φ^c and ψ^c should be replaced by $\varphi^c-\varphi^c_1$ and $\psi^c-\psi^c_2$, that is,

$$\begin{aligned}
\cos\theta^c &= \cos\theta^c_1\cos\theta^c_2-\sin\theta^c_1\sin\theta^c_2\cos(\varphi^c_2+\psi^c_1),\\
e^{i(\varphi^c-\varphi^c_1)} &= \frac{\sin\theta^c_1\cos\theta^c_2+\cos\theta^c_1\sin\theta^c_2\cos(\varphi^c_2+\psi^c_1)+i\sin\theta^c_2\sin(\varphi^c_2+\psi^c_1)}{\sin\theta^c},\\
e^{\frac{i(\varphi^c+\psi^c-\varphi^c_1-\psi^c_2)}{2}} &= \frac{\cos\frac{\theta^c_1}{2}\cos\frac{\theta^c_2}{2}e^{\frac{i(\varphi^c_2+\psi^c_1)}{2}}-\sin\frac{\theta^c_1}{2}\sin\frac{\theta^c_2}{2}e^{-\frac{i(\varphi^c_2+\psi^c_1)}{2}}}{\cos\dfrac{\theta^c}{2}}.
\end{aligned}\tag{71}$$

Addition Theorem for hyperspherical functions Z^l_{mn} follows from the relation

$$T_l(\mathfrak{g}_1\mathfrak{g}_2)=T_l(\mathfrak{g}_1)T_l(\mathfrak{g}_2).$$

Hence it follows that

$$t^l_{mn}(\mathfrak{g}_1\mathfrak{g}_2)=\sum_{k=-l}^{l}t^l_{mk}(\mathfrak{g}_1)t^l_{kn}(\mathfrak{g}_2).$$

Let us apply this equality to the matrices $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with the Euler angles $0,0,\theta_1,\tau_1,0,0$ and $\varphi_2,\epsilon_2,\theta_2,\tau_2,0,0$, respectively. Using the formula (47), we obtain

$$\begin{aligned}
t^l_{mk}(\mathfrak{g}_1) &= Z^l_{mk}(\cos\theta_1,\cosh\tau_1),\\
t^l_{kn}(\mathfrak{g}_2) &= e^{-k(\epsilon_2+i\varphi_2)}Z^l_{kn}(\cos\theta_2,\cosh\tau_2)
\end{aligned}$$

and

$$t^l_{mn}(\mathfrak{g}_1\mathfrak{g}_2)=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z^l_{mn}(\cos\theta,\cosh\tau),$$

where $\epsilon,\varphi,\theta,\tau,\varepsilon,\psi$ are the Euler angles of the matrix $\mathfrak{g}_1\mathfrak{g}_2$. In accordance with (67)–(69) these angles are expressed via the factor angles $0,0,\theta_1,\tau_1,0,0$ and $\varphi_2,\epsilon_2,\theta_2,\tau_2,0,0$. Thus, in this case the functions Z^l_{mn} satisfy the following addition theorem:

$$e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z^l_{mn}(\cos\theta^c)=\sum_{k=-l}^{l}e^{-k(\epsilon_2+i\varphi_2)}Z^l_{mk}(\cos\theta^c_1)Z^l_{kn}(\cos\theta^c_2).\tag{72}$$

In general case when the Euler angles are related by the formulae (71) we obtain

$$e^{-m[\epsilon+\epsilon_1+i(\varphi_1-\varphi)]-n[\varepsilon+\varepsilon_2-i(\psi_2-\psi)]}Z^l_{mn}(\cos\theta^c)=$$
$$\sum_{k=-l}^{l}e^{-k[\epsilon_2+\varepsilon_1+i(\varphi_2+\psi_1)]}Z^l_{mk}(\cos\theta^c_1)Z^l_{kn}(\cos\theta^c_2).$$

Addition theorems for the associated and zonal hyperspherical functions follow as particular cases from the proved theorem for the functions $Z^l_{mn}(\theta,\tau)$. In the subsections 3.4 and 3.5 these functions have been defined by the formulas

$$Z_l(\cos\theta^c)=Z^l_{00}(\cos\theta^c)$$

and

$$Z_l^m(\cos\theta^c) = Z_{m0}^l(\cos\theta^c).$$

Supposing $n = 0$ in the formula (72), we obtain an addition theorem for the associated hyperspherical functions:

$$e^{-m(\epsilon+i\varphi)}Z_l^m(\cos\theta^c) = \sum_{k=-l}^{l} e^{-k(\epsilon_2+i\varphi_2)}Z_{mk}^l(\cos\theta_1^c)Z_l^k(\cos\theta_2^c),$$

where the angles φ^c, φ_2^c, θ^c, θ_1^c, θ_2^c are related by (67)–(69). Supposing $m = 0$, $n = 0$, we obtain

$$Z_l(\cos\theta^c) = \sum_{k=-l}^{l} e^{-k(\epsilon_2+i\varphi_2)}Z_l^k(\cos\theta_1^c)Z_l^k(\cos\theta_2^c).$$

3.9 Matrix elements of the principal series representations of the Lorentz group

As it has been shown in [47], for the case of principal series representations there exists an analog of the spinor representation formula (32):

$$V_a f(z) = (a_{12}z + a_{22})^{\frac{\lambda}{2}+i\frac{\rho}{2}-1}\overline{(a_{12}z + a_{22})}^{-\frac{\lambda}{2}+i\frac{\rho}{2}-1} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right), \tag{73}$$

where $f(z)$ is a measurable functions of the Hilbert space $L_2(Z)$, satisfying the condition $\int |f(z)|^2 dz < \infty$, $z = x + iy$. At this point, the numbers l_0, l_1 and λ, ρ are related by the formulas

$$l_0 = \left|\frac{\lambda}{2}\right|, \quad l_1 = -i(\operatorname{sign}\lambda)\frac{\rho}{2} \quad \text{if } m \neq 0,$$
$$l_0 = 0, \quad l_1 = \pm i\frac{\rho}{2} \quad \text{if } m = 0.$$

A totality of all representations $a \rightarrow V_a$, corresponding to all possible pairs λ, ρ, is called a principal series of representations of the group $SL(2,\mathbb{C})$. At this point, a comparison of (73) with the formula (32) for the spinor representation $\mathfrak{S}_l$ shows that the both formulas have the same structure; only the exponents at the factors $(a_{12}z+a_{22})$, $\overline{(a_{12}z + a_{22})}$ and the functions $f(z)$ are different. In the case of spinor representations the functions $f(z)$ are polynomials $p(z,\bar{z})$ in the spaces $\mathrm{Sym}_{(k,r)}$, and in the case of a representation $\mathfrak{S}_{\lambda,\rho}$ of the principal series $f(z)$ are functions from the Hilbert space $L_2(Z)$.

As is known, a representation S_l of the group $SU(2)$ is realized in terms of the functions $P_{mn}^l(\cos\theta)$.

Theorem 1 (Naimark [47])**.** *The representation S_l is contained in $\mathfrak{S}_{\lambda,\rho}$ no more then one time. At this point, S_l is contained in $\mathfrak{S}_{\lambda,\rho}$, when $\frac{\lambda}{2}$ is one from the numbers $-l, -l+1, \dots, l$.*

Therefore, matrix elements of the principal series representations of the Lorentz group,

making infinite-dimensional matrix, have the form (see also [20]):

$$t^{-\frac{1}{2}+i\rho}_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z^{-\frac{1}{2}+i\rho}_{mn} = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times$$

$$\sum^{\left[\frac{\lambda}{2}\right]}_{k=-\frac{\lambda}{2}} i^{m-k}\sqrt{\Gamma(\frac{\lambda}{2}-m+1)\Gamma(\frac{\lambda}{2}+m+1)\Gamma(\frac{\lambda}{2}-k+1)\Gamma(\frac{\lambda}{2}+k+1)}\times$$

$$\cos^{\lambda}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times$$

$$\sum^{\left[\min\left(\frac{\lambda}{2}-m,\frac{\lambda}{2}+k\right)\right]}_{j=\max(0,k-m)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(\frac{\lambda}{2}-m-j+1)\Gamma(\frac{\lambda}{2}+k-j+1)\Gamma(m-k+j+1)}\times$$

$$\sqrt{\Gamma(\frac{1}{2}+i\rho-n)\Gamma(\frac{1}{2}+i\rho+n)\Gamma(\frac{1}{2}+i\rho-k)\Gamma(\frac{1}{2}+i\rho+k)}\cosh^{-1+2i\rho}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times$$

$$\sum^{\infty}_{s=\max(0,k-n)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(\frac{1}{2}+i\rho-n-s)\Gamma(\frac{1}{2}+i\rho+k-s)\Gamma(n-k+s+1)}. \quad (74)$$

Thus, the matrix elements of the principal series representations of the group $\mathfrak{G}_+$ are expressed via the function

$$\mathfrak{M}^{-\frac{1}{2}+i\rho}_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)}Z^{-\frac{1}{2}+i\rho}_{mn}(\cos\theta^c)e^{-n(\varepsilon+i\psi)}, \quad (75)$$

where

$$Z^{-\frac{1}{2}+i\rho}_{mn}(\cos\theta^c) = \sum^{\left[\frac{\lambda}{2}\right]}_{k=-\frac{\lambda}{2}} P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau).$$

In the case of associated functions ($n=0$) we obtain

$$Z^{m}_{-\frac{1}{2}+i\rho}(\cos\theta^c) = \sum^{\left[\frac{\lambda}{2}\right]}_{k=-\frac{\lambda}{2}} P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{k}_{-\frac{1}{2}+i\rho}(\cosh\tau), \quad (76)$$

where $\mathfrak{P}^{k}_{-\frac{1}{2}+i\rho}(\cosh\tau)$ are *conical functions* (see [7]). In this case our result agrees with the paper [20], where matrix elements (eigenfunctions of Casimir operators) of noncompact rotation groups are expressed in terms of conical and spherical functions (see also [76]).

Further, at $\lambda=0$ and $\rho=i\sigma$ from (73) it follows that

$$V_af(z) = |a_{12}z+a_{22}|^{-2-\sigma}f\left(\frac{a_{11}z+a_{21}}{a_{12}z+a_{22}}\right).$$

This formula defines an unitary representation $a\rightarrow V_a$ of supplementary series $\mathfrak{D}_\sigma$ of the group $SL(2,\mathbb{C})$. At this point, for the supplementary series the relations

$$l_0=0,\quad l_1=\pm\frac{\sigma}{2}$$

hold. In turn, the representation S_l of the group $SU(2)$ is contained in the representation $\mathfrak{D}_\sigma$ of supplementary series when l is an integer number. In this case S_l is contained in $\mathfrak{D}_\sigma$ exactly one time and the number $\frac{\lambda}{2}=0$ is one from the set $-l,-l+1,\ldots,l$ [47].

Thus, matrix elements of supplementary series appear as a particular case of the matrix elements of principal series at $\lambda=0$ and $\rho=i\sigma$:

$$t^{-\frac{1}{2}-\sigma}_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z^{-\frac{1}{2}-\sigma}_{mn}=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times$$
$$\sqrt{\Gamma(\tfrac{1}{2}-\sigma-n)\Gamma(\tfrac{1}{2}-\sigma+n)\Gamma(\tfrac{1}{2}-\sigma-k)\Gamma(\tfrac{1}{2}-\sigma+k)}\cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$$\sum^{\infty}_{s=\max(0,k-n)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(\frac{1}{2}-\sigma-n-s)\Gamma(\frac{1}{2}-\sigma+k-s)\Gamma(n-k+s+1)}. \qquad (77)$$

Or

$$\mathfrak{M}^{-\frac{1}{2}-\sigma}_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}\mathfrak{P}^{-\frac{1}{2}-\sigma}_{mn}(\cosh\tau)e^{-n(\varepsilon+i\psi)},$$

that is, the hyperspherical function $Z^{-\frac{1}{2}+i\rho}_{mn}(\cos\theta^c)$ in the case of supplementary series is degenerated to the Jacobi function $\mathfrak{P}^{-\frac{1}{2}-\sigma}_{mn}(\cosh\tau)$. For the associated functions of supplementary series we obtain

$$\mathfrak{P}^m_{-\frac{1}{2}-\sigma}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}\mathfrak{P}^m_{-\frac{1}{2}-\sigma}(\cosh\tau).$$

4 Infinitesimal operators of $SU(2)\otimes SU(2)$

Let $\omega^c(t)$ be the one–parameter subgroup of $SL(2,\mathbb{C})$. The operators of the right regular representation of $SU(2)\otimes SU(2)$, corresponded to the elements of this subgroup, transfer complex functions $f(\mathfrak{g})$ into $R(\omega^c(t))f(\mathfrak{g})=f(\mathfrak{g}\omega^c(t))$. By this reason the infinitesimal operator of the right regular representation $R(\mathfrak{g})$, associated with one–parameter subgroup $\omega^c(t)$, transfers the function $f(\mathfrak{g})$ into $\frac{df(\mathfrak{g}\omega^c(t))}{dt}$ at $t=0$.

Let us denote Euler angles of the element $\mathfrak{g}\omega^c(t)$ via $\varphi^c(t),\theta^c(t),\psi^c(t)$. Then there is an equality

$$\left.\frac{df(\mathfrak{g}\omega^c(t))}{dt}\right|_{t=0}=\frac{\partial f}{\partial\varphi^c}\left(\varphi^c(0)\right)'+\frac{\partial f}{\partial\theta^c}\left(\theta^c(0)\right)'+\frac{\partial f}{\partial\psi^c}\left(\psi^c(0)\right)'.$$

The infinitesimal operator A^c_ω, corresponded to the subgroup $\omega^c(t)$, has a form

$$\mathsf{A}^c_\omega=\mathsf{A}_\omega-i\mathsf{B}_\omega=\left(\varphi^c(0)\right)'\frac{\partial}{\partial\varphi^c}+\left(\theta^c(0)\right)'\frac{\partial}{\partial\theta^c}+\left(\psi^c(0)\right)'\frac{\partial}{\partial\psi^c}.$$

Let us calculate infinitesimal operators A^c_1, A^c_2, A^c_3 corresponding the complex subgroups Ω^c_1, Ω^c_2, Ω^c_3. The subgroup Ω^c_3 consists of the matrices

$$\omega_3(t^c)=\begin{pmatrix} e^{\frac{it^c}{2}} & 0\\ 0 & e^{-\frac{it^c}{2}}\end{pmatrix}.$$

Let $\mathfrak{g}=\mathfrak{g}(\varphi^c,\theta^c,\psi^c)$ be a matrix with complex Euler angles $\varphi^c=\varphi-i\epsilon$, $\theta^c=\theta-i\tau$, $\psi^c=\psi-i\varepsilon$. Therefore, Euler angles of the matrix $\mathfrak{g}\omega_3(t^c)$ equal to φ^c, θ^c, ψ^c+t-it. Hence it follows that

$$\varphi'(0)=0,\quad \epsilon'(0)=0,\quad \theta'(0)=0,\quad \tau'(0)=0,\quad \psi'(0)=1,\quad \varepsilon'(0)=-i.$$

So, the operator A_3^c, corresponded to the subgroup Ω_3^c, has a form

$$\mathsf{A}_3^c = \frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\varepsilon}.$$

Whence

$$\mathsf{A}_3 = \frac{\partial}{\partial\psi}, \tag{78}$$

$$\mathsf{B}_3 = \frac{\partial}{\partial\varepsilon}. \tag{79}$$

Let us calculate the infinitesimal operator A_1^c corresponded the complex subgroup Ω_1^c. The subgroup Ω_1^c consists of the following matrices

$$\omega_1(t^c) = \begin{pmatrix} \cos\frac{t^c}{2} & i\sin\frac{t^c}{2} \\ i\sin\frac{t^c}{2} & \cos\frac{t^c}{2} \end{pmatrix}.$$

The Euler angles of these matrices equal to 0, $t^c = t - it$, 0. Let us represent the matrix $\mathfrak{g}\omega_1(t^c)$ by the product $\mathfrak{g}_1\mathfrak{g}_2$, the Euler angles of which are described by the formulae (71). Then the Euler angles of the matrix $\omega_1(t^c)$ equal to $\varphi_2^c = 0$, $\theta_2^c = t - it$, $\psi_2^c = 0$, and the Euler angles of the matrix $\mathfrak{g}$ equal to $\varphi_1^c = \varphi^c$, $\theta_1^c = \theta^c$, $\psi_1^c = \psi^c$. Thus, from the general formulae (71) we obtain that Euler angles $\varphi^c(t)$, $\theta^c(t)$, $\psi^c(t)$ of the matrix $\mathfrak{g}\omega_1(t^c)$ are defined by the following relations:

$$\cos\theta^c(t) = \cos\theta^c\cos t^c - \sin\theta^c\sin t^c\cos\psi^c, \tag{80}$$

$$e^{i\varphi^c(t)} = e^{i\varphi^c}\frac{\sin\theta^c\cos t^c + \cos\theta^c\sin t^c\cos\psi^c + i\sin t^c\sin\psi^c}{\sin\theta^c(t)}, \tag{81}$$

$$e^{\frac{i[\varphi^c(t)+\psi^c(t)]}{2}} = e^{\frac{i\varphi^c}{2}}\frac{\cos\frac{\theta^c}{2}\cos\frac{t^c}{2}e^{\frac{i\psi^c}{2}} - \sin\frac{\theta^c}{2}\sin\frac{t^c}{2}e^{-\frac{i\psi^c}{2}}}{\cos\frac{\theta^c(t)}{2}}. \tag{82}$$

For calculation of derivatives $\varphi'(t)$, $\epsilon'(t)$, $\theta'(t)$, $\tau'(t)$, $\psi'(t)$, $\varepsilon'(t)$ at $t = 0$ we differentiate on t the both parts of the each equality from (80)–(82) and take $t = 0$. At this point we have $\varphi(0) = \varphi$, $\epsilon(0) = \epsilon$, $\theta(0) = \theta$, $\tau(0) = \tau$, $\psi(0) = \psi$, $\varepsilon(0) = \varepsilon$.

So, let us differentiate the both parts of (80). In the result we obtain

$$\theta'(t) - i\tau'(t) = (1 - i)\cos\psi^c.$$

Taking $t = 0$, we find that

$$\theta'(0) = \cos\psi^c, \quad \tau'(0) = \cos\psi^c.$$

Differentiating now the both parts of (81), we obtain

$$\varphi'(0) - i\epsilon'(0) = \frac{(1-i)\sin\psi^c}{\sin\theta^c}.$$

Therefore,

$$\varphi'(0) = \frac{\sin\psi^c}{\sin\theta^c}, \quad \epsilon'(0) = \frac{\sin\psi^c}{\sin\theta^c}.$$

Further, differentiating the both parts of (82), we find that

$$\psi'(0) - i\varepsilon'(0) = -(1+i)\cot\theta^c \sin\psi^c$$

and

$$\psi'(0) = -\cot\theta^c \sin\psi^c, \quad \varepsilon'(0) = -\cot\theta^c \sin\psi^c.$$

In such a way, we obtain the following infinitesimal operators:

$$\mathsf{A}_1 = \cos\psi^c \frac{\partial}{\partial\theta} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi} - \cot\theta^c \sin\psi^c \frac{\partial}{\partial\psi}, \tag{83}$$

$$\mathsf{B}_1 = \cos\psi^c \frac{\partial}{\partial\tau} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c \sin\psi^c \frac{\partial}{\partial\varepsilon}. \tag{84}$$

Let us calculate now an infinitesimal operator A_2^c corresponded to the complex subgroup Ω_2^c. The subgroup Ω_2^c consists of the following matrices

$$\omega_2(t^c) = \begin{pmatrix} \cos\frac{t^c}{2} & -\sin\frac{t^c}{2} \\ \sin\frac{t^c}{2} & \cos\frac{t^c}{2} \end{pmatrix},$$

where the Euler angles equal correspondingly to 0, $t^c = t - it$, 0. It is obvious that the matrix $\mathfrak{g}\omega_2(t^c)$ can be represented by the product

$$\mathfrak{g}_1\mathfrak{g}_2 = \begin{pmatrix} \cos\frac{\theta_1^c}{2} & i\sin\frac{\theta_1^c}{2} \\ i\sin\frac{\theta_1^c}{2} & \cos\frac{\theta_1^c}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_2^c}{2}e^{\frac{i\varphi_2^c}{2}} & -\sin\frac{\theta_2^c}{2}e^{\frac{i\varphi_2^c}{2}} \\ \sin\frac{\theta_2^c}{2}e^{-\frac{i\varphi_2^c}{2}} & \cos\frac{\theta_2^c}{2}e^{-\frac{i\varphi_2^c}{2}} \end{pmatrix}.$$

Multiplying the matrices in the right part of this equality, we obtain that Euler angles of the product $\mathfrak{g}_1\mathfrak{g}_2$ are related by the formulae

$$\cos\theta^c = \cos\theta_1^c\cos\theta_2^c + \sin\theta_1^c\sin\theta_2^c\sin\varphi_2^c, \tag{85}$$

$$e^{i\varphi^c} = \frac{\sin\theta_1^c\cos\theta_2^c - \cos\theta_1^c\sin\theta_2^c\sin\varphi_2^c + i\sin\theta_2^c\cos\varphi_2^c}{\sin\theta^c}, \tag{86}$$

$$e^{\frac{i(\varphi^c+\psi^c)}{2}} = \frac{\cos\frac{\theta_1^c}{2}\cos\frac{\theta_2^c}{2}e^{\frac{i\varphi_2^c}{2}} + i\sin\frac{\theta_1^c}{2}\sin\frac{\theta_2^c}{2}e^{-\frac{i\varphi_2^c}{2}}}{\cos\frac{\theta^c}{2}}. \tag{87}$$

Or, repeating the calculations as in the case of (71), we obtain in general case

$$\begin{aligned} \cos\theta^c &= \cos\theta_1^c\cos\theta_2^c + \sin\theta_1^c\sin\theta_2^c\sin(\varphi_2^c+\psi_1^c), \\ e^{i(\varphi^c-\varphi_1^c)} &= \frac{\sin\theta_1^c\cos\theta_2^c - \cos\theta_1^c\sin\theta_2^c\sin(\varphi_2^c+\psi_1^c) + i\sin\theta_2^c\cos(\varphi_2^c+\psi_1^c)}{\sin\theta^c}, \\ e^{\frac{i(\varphi^c+\psi^c-\varphi_1^c-\psi_2^c)}{2}} &= \frac{\cos\frac{\theta_1^c}{2}\cos\frac{\theta_2^c}{2}e^{\frac{i(\varphi_2^c+\psi_1^c)}{2}} + \sin\frac{\theta_1^c}{2}\sin\frac{\theta_2^c}{2}e^{-\frac{i(\varphi_2^c+\psi_1^c)}{2}}}{\cos\frac{\theta^c}{2}}. \end{aligned} \tag{88}$$

Therefore, Euler angles of the matrix $\omega_2(t^c)$ equal to $\varphi_2^c = 0$, $\theta_2^c = t - it$, $\psi_2^c = 0$, and Euler angles of the matrix $\mathfrak{g}$ equal to $\varphi_1^c = \varphi^c$, $\theta_1^c = \theta^c$, $\psi_1^c = \psi^c$. Then from the formulae (88) we

obtain that Euler angles $\varphi^c(t)$, $\theta^c(t)$, $\psi^c(t)$ of the matrix $\mathfrak{g}\omega_2(t^c)$ are defined by relations

$$\cos\theta^c(t) = \cos\theta^c\cos t^c + \sin\theta^c\sin t^c\sin\psi^c, \tag{89}$$

$$e^{i\varphi^c(t)} = e^{i\varphi^c}\frac{\sin\theta^c\cos t^c - \cos\theta^c\sin t^c\sin\psi^c + i\sin t^c\cos\psi^c}{\sin\theta^c(t)}, \tag{90}$$

$$e^{\frac{i[\varphi^c(t)+\psi^c(t)]}{2}} = e^{\frac{i\varphi^c}{2}}\frac{\cos\frac{\theta^c}{2}\cos\frac{t^c}{2}e^{\frac{i\psi^c}{2}} + i\sin\frac{\theta^c}{2}\sin\frac{t^c}{2}e^{-\frac{i\psi^c}{2}}}{\cos\frac{\theta^c(t)}{2}}. \tag{91}$$

Differentiating on t the both parts of the each equalities (89)–(91) and taking $t=0$, we obtain

$$\begin{aligned}
&\theta'(0) = \tau'(0) = -\sin\psi^c,\\
&\varphi'(0) = \epsilon'(0) = \frac{\cos\psi^c}{\sin\theta^c},\\
&\psi'(0) = \varepsilon'(0) = -\cot\theta^c\cos\psi^c.
\end{aligned} \tag{92}$$

Therefore, for the subgroup Ω_2^c we have the following infinitesimal operators:

$$\mathsf{A}_2 = -\sin\psi^c\frac{\partial}{\partial\theta} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi} - \cot\theta^c\cos\psi^c\frac{\partial}{\partial\psi}, \tag{93}$$

$$\mathsf{B}_2 = -\sin\psi^c\frac{\partial}{\partial\tau} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c\cos\psi^c\frac{\partial}{\partial\varepsilon}. \tag{94}$$

It is easy to verify that operators A_i, B_i, defined by the formulae (78), (79), (83), (84) and (93), (94), are satisfy the commutation relations (4).

4.1 Casimir operators and differential equations for hyperspherical functions

Taking into account the expressions (78), (79), (83), (84) and (93), (94) we can write the operators (5) in the form

$$\mathsf{X}_1 = \cos\psi^c\frac{\partial}{\partial\theta^c} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi^c} - \cot\theta^c\sin\psi^c\frac{\partial}{\partial\psi^c}, \tag{95}$$

$$\mathsf{X}_2 = -\sin\psi^c\frac{\partial}{\partial\theta^c} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi^c} - \cot\theta^c\cos\psi^c\frac{\partial}{\partial\psi^c}, \tag{96}$$

$$\mathsf{X}_3 = \frac{\partial}{\partial\psi^c}, \tag{97}$$

$$\mathsf{Y}_1 = \cos\dot{\psi}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{\sin\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c} - \cot\dot{\theta}^c\sin\dot{\psi}^c\frac{\partial}{\partial\dot{\psi}^c}, \tag{98}$$

$$\mathsf{Y}_2 = -\sin\dot{\psi}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{\cos\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c} - \cot\dot{\theta}^c\cos\dot{\psi}^c\frac{\partial}{\partial\dot{\psi}^c}, \tag{99}$$

$$\mathsf{Y}_3 = \frac{\partial}{\partial\dot{\psi}^c}, \tag{100}$$

where

$$\frac{\partial}{\partial\theta^c}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial\tau}\right),\quad \frac{\partial}{\partial\varphi^c}=\frac{1}{2}\left(\frac{\partial}{\partial\varphi}+i\frac{\partial}{\partial\epsilon}\right),\quad \frac{\partial}{\partial\psi^c}=\frac{1}{2}\left(\frac{\partial}{\partial\psi}+i\frac{\partial}{\partial\varepsilon}\right),$$
$$\frac{\partial}{\partial\dot{\theta}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}-i\frac{\partial}{\partial\tau}\right),\quad \frac{\partial}{\partial\dot{\varphi}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\varphi}-i\frac{\partial}{\partial\epsilon}\right),\quad \frac{\partial}{\partial\dot{\psi}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\psi}-i\frac{\partial}{\partial\varepsilon}\right),$$

As is known, for the Lorentz group there are two independent Casimir operators

$$\begin{aligned}\mathsf{X}^2 &= \mathsf{X}_1^2+\mathsf{X}_2^2+\mathsf{X}_3^2=\frac{1}{4}(\mathsf{A}^2-\mathsf{B}^2+2i\mathsf{AB}),\\ \mathsf{Y}^2 &= \mathsf{Y}_1^2+\mathsf{Y}_2^2+\mathsf{Y}_3^2=\frac{1}{4}(\widetilde{\mathsf{A}}^2-\widetilde{\mathsf{B}}^2-2i\widetilde{\mathsf{A}}\widetilde{\mathsf{B}}).\end{aligned}\tag{101}$$

Substituting (95)-(100) into (101), we obtain for the Casimir operators the following expressions

$$\begin{aligned}\mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}}+\cot\theta^c\frac{\partial}{\partial\theta^c}+\frac{1}{\sin^2\theta^c}\left[\frac{\partial^2}{\partial\varphi^{c2}}-2\cos\theta^c\frac{\partial}{\partial\varphi^c}\frac{\partial}{\partial\psi^c}+\frac{\partial^2}{\partial\psi^{c2}}\right],\\ \mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}}+\cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c}+\frac{1}{\sin^2\dot{\theta}^c}\left[\frac{\partial^2}{\partial\dot{\varphi}^{c2}}-2\cos\dot{\theta}^c\frac{\partial}{\partial\dot{\varphi}^c}\frac{\partial}{\partial\dot{\psi}^c}+\frac{\partial^2}{\partial\dot{\psi}^{c2}}\right].\end{aligned}\tag{102}$$

Matrix elements of unitary irreducible representations of the Lorentz group are eigenfunctions of the operators (102):

$$\begin{aligned}\left[\mathsf{X}^2+l(l+1)\right]\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c) &= 0,\\ \left[\mathsf{Y}^2+\dot{l}(\dot{l}+1)\right]\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,\dot{\psi}^c) &= 0,\end{aligned}\tag{103}$$

where

$$\begin{aligned}\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c) &= e^{-i(m\varphi^c+n\psi^c)}Z^l_{mn}(\theta^c),\\ \mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,\dot{\psi}^c) &= e^{-i(\dot{m}\dot{\varphi}^c+\dot{n}\dot{\varphi}^c)}Z^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\theta}^c).\end{aligned}\tag{104}$$

Substituting the hyperspherical functions (104) into (103) and taking into account the operators (102), we obtain

$$\begin{aligned}\left[\frac{d^2}{d\theta^{c2}}+\cot\theta^c\frac{d}{d\theta^c}-\frac{m^2+n^2-2mn\cos\theta^c}{\sin^2\theta^c}+l(l+1)\right]Z^l_{mn}(\theta^c) &= 0,\\ \left[\frac{d^2}{d\dot{\theta}^{c2}}+\cot\dot{\theta}^c\frac{d}{d\dot{\theta}^c}-\frac{\dot{m}^2+\dot{n}^2-2\dot{m}\dot{n}\cos\dot{\theta}^c}{\sin^2\dot{\theta}^c}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\theta}^c) &= 0.\end{aligned}$$

Finally, after substitutions $z=\cos\theta^c$ and $\overset{*}{z}=\cos\dot{\theta}^c$, we come to the following differential equations

$$\begin{aligned}\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2}+l(l+1)\right]Z^l_{mn}(\arccos z) &= 0,\\ \left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2}-2\overset{*}{z}\frac{d}{d\overset{*}{z}}-\frac{\dot{m}^2+\dot{n}^2-2\dot{m}\dot{n}\overset{*}{z}}{1-\overset{*}{z}{}^2}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\arccos\overset{*}{z}) &= 0.\end{aligned}$$

The latter equations have three singular points -1, $+1$, ∞.

Further, Casimir operators on the 2-dimensional complex sphere (correspondingly, on the dual sphere) have the form

$$
\begin{aligned}
\mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}} + \cot\theta^c\frac{\partial}{\partial\theta^c} + \frac{1}{\sin^2\theta^c}\frac{\partial^2}{\partial\varphi^{c2}},\\
\mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{1}{\sin^2\dot{\theta}^{c2}}\frac{\partial^2}{\partial\dot{\varphi}^{c2}}.
\end{aligned}
\tag{105}
$$

Associated hyperspherical functions $\mathfrak{M}_l^m(\varphi^c,\theta^c,0)$ and $\mathfrak{M}_{\dot{l}}^{\dot{m}}(\dot{\varphi}^c,\dot{\theta}^c,0)$, defined on the surface of the two-dimensional complex sphere, are eigenfunctions of the operators X^2 and Y^2:

$$
\begin{aligned}
\left[\mathsf{X}^2 + l(l+1)\right]\mathfrak{M}_l^m(\varphi^c,\theta^c,0) &= 0,\\
\left[\mathsf{Y}^2 + \dot{l}(\dot{l}+1)\right]\mathfrak{M}_{\dot{l}}^{\dot{m}}(\dot{\varphi}^c,\dot{\theta}^c,0) &= 0
\end{aligned}
\tag{106}
$$

Substituting the functions $\mathfrak{M}_l^m(\varphi^c,\theta^c,0)=e^{-im\varphi^c}Z_l^m(\theta^c)$, $\mathfrak{M}_{\dot{l}}^{\dot{m}}(\dot{\varphi}^c,\dot{\theta}^c,0)=e^{-i\dot{m}\dot{\varphi}^c}Z_{\dot{l}}^{\dot{m}}(\dot{\theta}^c)$ into (106) and taking into account the operators (105), we obtain the following equations

$$
\begin{aligned}
\left[\frac{d^2}{d\theta^{c2}} + \cot\theta^c\frac{d}{d\theta^c} - \frac{m^2}{\sin^2\theta^c} + l(l+1)\right]Z_l^m(\theta^c) &= 0,\\
\left[\frac{d^2}{d\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{d}{d\dot{\theta}^c} - \frac{m^2}{\sin^2\dot{\theta}^c} + \dot{l}(\dot{l}+1)\right]Z_{\dot{l}}^{\dot{m}}(\dot{\theta}^c) &= 0.
\end{aligned}
$$

Or, introducing the substitutions $z=\cos\theta^c$, $\overset{*}{z}=\cos\dot{\theta}^c$, we find that

$$
\begin{aligned}
\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2}{1-z^2} + l(l+1)\right]Z_l^m(\arccos z) &= 0,\\
\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2} - 2\overset{*}{z}\frac{d}{d\overset{*}{z}} - \frac{\dot{m}^2}{1-\overset{*}{z}{}^2} + \dot{l}(\dot{l}+1)\right]Z_{\dot{l}}^{\dot{m}}(\arccos\overset{*}{z}) &= 0.
\end{aligned}
$$

Analogously, Casimir operators, corresponding to zonal hyperspherical functions $Z_l(\theta^c)$, have the form

$$
\begin{aligned}
\mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}} + \cot\theta^c\frac{\partial}{\partial\theta^c},\\
\mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c}.
\end{aligned}
\tag{107}
$$

And the equations for Z_l are

$$
\begin{aligned}
\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + l(l+1)\right]Z_l(\arccos z) &= 0,\\
\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2} - 2\overset{*}{z}\frac{d}{d\overset{*}{z}} + \dot{l}(\dot{l}+1)\right]Z_{\dot{l}}(\arccos\overset{*}{z}) &= 0.
\end{aligned}
$$

5 Recurrence relations between hyperspherical functions

Between generalized hyperspherical functions $\mathfrak{M}^l_{mn}$ (and also the hyperspherical functions Z^l_{mn}) there exists a wide variety of recurrence relations. Part of them relates the hyperspherical functions of one and the same order (with identical l), other part relates the functions of different orders.

In virtue of the Van der Waerden representation (8) the recurrence formulae for the hyperspherical functions of one and the same order follow from the equalities

$$\mathsf{X}_-\mathfrak{M}^l_{mn} = \boldsymbol{\alpha}_n\mathfrak{M}^l_{m,n-1}, \quad \mathsf{X}_+\mathfrak{M}^l_{mn} = \boldsymbol{\alpha}_{n+1}\mathfrak{M}^l_{m,n+1}, \tag{108}$$

$$\mathsf{Y}_-\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}} = \boldsymbol{\alpha}_{\dot{n}}\mathfrak{M}^{\dot{l}}_{\dot{m},\dot{n}-1}, \quad \mathsf{Y}_+\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}} = \boldsymbol{\alpha}_{\dot{n}+1}\mathfrak{M}^{\dot{l}}_{\dot{m},\dot{n}+1}, \tag{109}$$

where

$$\boldsymbol{\alpha}_n = \sqrt{(l+n)(l-n+1)}, \quad \boldsymbol{\alpha}_{\dot{n}} = \sqrt{(\dot{l}+\dot{n})(\dot{l}-\dot{n}+1)}.$$

From (5) and (7) it follows that

$$\begin{aligned}
\mathsf{X}_+ &= \frac{1}{2}\left(i\mathsf{A}_1 - \mathsf{A}_2 - \mathsf{B}_1 - i\mathsf{B}_2\right),\\
\mathsf{X}_- &= \frac{1}{2}\left(i\mathsf{A}_1 + \mathsf{A}_2 - \mathsf{B}_1 + i\mathsf{B}_2\right),\\
\mathsf{Y}_+ &= \frac{1}{2}\left(i\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2 + \widetilde{\mathsf{B}}_1 + i\widetilde{\mathsf{B}}_2\right),\\
\mathsf{Y}_- &= \frac{1}{2}\left(i\widetilde{\mathsf{A}}_1 + \widetilde{\mathsf{A}}_2 + \widetilde{\mathsf{B}}_1 - i\widetilde{\mathsf{B}}_2\right).
\end{aligned} \tag{110}$$

Using the formulae (83), (84) and (93), (94), we obtain

$$\mathsf{X}_+ = \frac{e^{-i\psi^c}}{2}\left[i\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta^c}\frac{\partial}{\partial\varphi} + \cot\theta^c\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\tau} - \frac{i}{\sin\theta^c}\frac{\partial}{\partial\epsilon} + i\cot\theta^c\frac{\partial}{\partial\varepsilon}\right], \tag{111}$$

$$\mathsf{X}_- = \frac{e^{i\psi^c}}{2}\left[i\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta^c}\frac{\partial}{\partial\varphi} - \cot\theta^c\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\tau} + \frac{i}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - i\cot\theta^c\frac{\partial}{\partial\varepsilon}\right], \tag{112}$$

$$\mathsf{Y}_+ = \frac{e^{-i\dot{\psi}^c}}{2}\left[i\frac{\partial}{\partial\theta} - \frac{1}{\sin\dot{\theta}^c}\frac{\partial}{\partial\varphi} + \cot\dot{\theta}^c\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\tau} + \frac{i}{\sin\dot{\theta}^c}\frac{\partial}{\partial\epsilon} - i\cot\dot{\theta}^c\frac{\partial}{\partial\varepsilon}\right], \tag{113}$$

$$\mathsf{Y}_- = \frac{e^{i\dot{\psi}^c}}{2}\left[i\frac{\partial}{\partial\theta} + \frac{1}{\sin\dot{\theta}^c}\frac{\partial}{\partial\varphi} - \cot\dot{\theta}^c\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\tau} - \frac{i}{\sin\dot{\theta}^c}\frac{\partial}{\partial\epsilon} + i\cot\dot{\theta}^c\frac{\partial}{\partial\varepsilon}\right]. \tag{114}$$

Further, substituting the function $\mathfrak{M}^l_{mn} = e^{-m(\epsilon-i\varphi)}Z^l_{mn}(\theta,\tau)e^{-n(\varepsilon-i\psi)}$ into the relations (108) and taking into account the operators (111) and (112), we find that

$$i\frac{\partial \dot{Z}^l_{mn}}{\partial\theta} - \frac{\partial \dot{Z}^l_{mn}}{\partial\tau} - \frac{2i(m-n\cos\theta^c)}{\sin\theta^c}\dot{Z}^l_{mn} = 2\boldsymbol{\alpha}_n\dot{Z}^l_{m,n-1}, \tag{115}$$

$$i\frac{\partial \dot{Z}^l_{mn}}{\partial\theta} - \frac{\partial \dot{Z}^l_{mn}}{\partial\tau} + \frac{2i(m-n\cos\theta^c)}{\sin\theta^c}\dot{Z}^l_{mn} = 2\boldsymbol{\alpha}_{n+1}\dot{Z}^l_{m,n+1}. \tag{116}$$

Since the functions $\dot{Z}^l_{mn}(\theta,\tau)$ are symmetric, that is, $\dot{Z}^l_{mn}(\theta,\tau) = \dot{Z}^l_{nm}(\theta,\tau)$, then substituting $\dot{Z}^l_{nm}(\theta,\tau)$ in lieu of $\dot{Z}^l_{mn}$ into the formulae (115)–(116) and replacing m by n, and n by m,

we obtain

$$
\begin{aligned}
i\frac{\partial Z^l_{mn}}{\partial\theta}-\frac{\partial Z^l_{mn}}{\partial\tau}-\frac{2i(n-m\cos\theta^c)}{\sin\theta^c}Z^l_{mn} &= 2\boldsymbol{\alpha}_m Z^l_{m-1,n}, && (117)\\
i\frac{\partial Z^l_{mn}}{\partial\theta}-\frac{\partial Z^l_{mn}}{\partial\tau}+\frac{2i(n-m\cos\theta^c)}{\sin\theta^c}Z^l_{mn} &= 2\boldsymbol{\alpha}_{m+1} Z^l_{m+1,n}. && (118)
\end{aligned}
$$

Analogously, substituting the function $\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}=e^{-\dot{m}(\epsilon-i\varphi)}Z^{\dot{l}}_{\dot{m}\dot{n}}(\theta,\tau)e^{-\dot{n}(\varepsilon-i\psi)}$ into the relations (109), we obtain

$$
\begin{aligned}
i\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\theta}+\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\tau}+\frac{2i(\dot{m}-\dot{n}\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}}_{\dot{m}\dot{n}} &= 2\boldsymbol{\alpha}_{\dot{n}} Z^{\dot{l}}_{\dot{m},\dot{n}-1}, && (119)\\
i\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\theta}+\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\tau}-\frac{2i(\dot{m}-\dot{n}\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}}_{\dot{m}\dot{n}} &= 2\boldsymbol{\alpha}_{\dot{n}+1} Z^{\dot{l}}_{\dot{m},\dot{n}+1}. && (120)
\end{aligned}
$$

Further, using the symmetry of the functions $Z^{\dot{l}}_{\dot{m}\dot{n}}$, we obtain

$$
\begin{aligned}
i\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\theta}+\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\tau}+\frac{2i(\dot{n}-\dot{m}\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}}_{\dot{m}\dot{n}} &= 2\boldsymbol{\alpha}_{\dot{m}} Z^{\dot{l}}_{\dot{m}-1,\dot{n}}, && (121)\\
i\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\theta}+\frac{\partial Z^{\dot{l}}_{\dot{m}\dot{n}}}{\partial\tau}-\frac{2i(\dot{n}-\dot{m}\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}}_{\dot{m}\dot{n}} &= 2\boldsymbol{\alpha}_{\dot{m}+1} Z^{\dot{l}}_{\dot{m}+1,\dot{n}}. && (122)
\end{aligned}
$$

Supposing $n=0$ ($\dot{n}=0$) in the formulas (117)–(118) and (121)–(122), we obtain recurrence relations for the associated hyperspherical functions:

$$
\begin{aligned}
i\frac{\partial Z^m_l}{\partial\theta}-\frac{\partial Z^m_l}{\partial\tau}+2im\cot\theta^c Z^m_l &= 2\boldsymbol{\alpha}_m Z^{m-1}_l,\\
i\frac{\partial Z^m_l}{\partial\theta}-\frac{\partial Z^m_l}{\partial\tau}-2im\cot\theta^c Z^m_l &= 2\boldsymbol{\alpha}_{m+1} Z^{m+1}_l,
\end{aligned}
$$

and

$$
\begin{aligned}
i\frac{\partial Z^{\dot{m}}_{\dot{l}}}{\partial\theta}+\frac{\partial Z^{\dot{m}}_{\dot{l}}}{\partial\tau}-2i\dot{m}\cot\dot{\theta}^c Z^{\dot{m}}_{\dot{l}} &= 2\boldsymbol{\alpha}_{\dot{m}} Z^{\dot{m}-1}_{\dot{l}},\\
i\frac{\partial Z^{\dot{m}}_{\dot{l}}}{\partial\theta}+\frac{\partial Z^{\dot{m}}_{\dot{l}}}{\partial\tau}+2i\dot{m}\cot\dot{\theta}^c Z^{\dot{m}}_{\dot{l}} &= 2\boldsymbol{\alpha}_{\dot{m}+1} Z^{\dot{m}+1}_{\dot{l}}.
\end{aligned}
$$

Let us consider now recurrence relations between hyperspherical functions with different order. These recurrence formulae are related with the tensor products of irreducible representations of the Lorentz group. Indeed, in accordance with Van der Waerden representation (8) an arbitrary finite–dimensional representation of the group $\mathfrak{G}_+$ has a form $\boldsymbol{\tau}_{l0}\otimes\boldsymbol{\tau}_{0\dot{l}}\sim\boldsymbol{\tau}_{l\dot{l}}$, where $\boldsymbol{\tau}_{l0}$ and $\boldsymbol{\tau}_{0\dot{l}}$ are representations of the group $SU(2)$. Then a product of the two representations $\boldsymbol{\tau}_{l_1\dot{l}_1}$ and $\boldsymbol{\tau}_{l_2\dot{l}_2}$ of the Lorentz group is defined by an expression

$$
\boldsymbol{\tau}_{l_1\dot{l}_1}\otimes\boldsymbol{\tau}_{l_2\dot{l}_2}=\sum_{|l_1-l_2|\le m\le l_1+l_2;|\dot{l}_1-\dot{l}_2|\le\dot{m}\le\dot{l}_1+\dot{l}_2}\boldsymbol{\tau}_{m\dot{m}}. \tag{123}
$$

The vectors $\zeta_{lm;\dot{l}\dot{m}}=\mid lm;\dot{l}\dot{m}\rangle$ of the helicity basis have the form

$$
\zeta_{lm;\dot{l}\dot{m}}=\sum_{\substack{m_1+m_2=m,\\ \dot{m}_1+\dot{m}_2=\dot{m}}}C(l_1,l_2,l;m_1,m_2,m)C(\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m})\zeta_{l_1m_1;\dot{l}_1\dot{m}_1}\otimes\zeta_{l_2m_2;\dot{l}_2\dot{m}_2}, \tag{124}
$$

where

$$C(l_1,l_2,l;m_1,m_2,m)C(\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m})=B^{l_1,l_2,l;m_1,m_2,m}_{\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m}}$$

are the Clebsch–Gordan coefficients of the group $SL(2,\mathbb{C})$. Expressing the Clebsch–Gordan coefficients $C(l_1,l_2,l;m_1,m_2,m_1+m_2)$ of the group $SU(2)$ via a generalized hypergeometric function ${}_3F_2$ (see, for example, [51, 77, 74]), we see that CG–coefficients of $SL(2,\mathbb{C})$ have the form

$$\begin{gathered}
B^{l_1,l_2,l;m_1,m_2,m}_{\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m}}=(-1)^{l_1+\dot{l}_1-m_1-\dot{m}_1}\times\\
\frac{\Gamma(l_1+l_2-m+1)\Gamma(\dot{l}_1+\dot{l}_2-\dot{m}+1)}{\Gamma(l_2-l_1+m+1)\Gamma(\dot{l}_2-\dot{l}_1+\dot{m}+1)}\times\\
\left(\frac{(l-m)!(l+l_2-l_1)!(l_1-m_1)!(l_2+m_2)!(l+m)!(2l+1)}{(l_1-l_2+l)!(l_1+l_2-l)!(l_1+l_2+l)!(l_1-m_1)!(l_2-m_2)!}\right)^{\frac{1}{2}}\times\\
\left(\frac{(\dot{l}-\dot{m})!(\dot{l}+\dot{l}_2-\dot{l}_1)!(\dot{l}_1-\dot{m}_1)!(\dot{l}_2+\dot{m}_2)!(\dot{l}+\dot{m})!(2\dot{l}+1)}{(\dot{l}_1-\dot{l}_2+\dot{l})!(\dot{l}_1+\dot{l}_2-\dot{l})!(\dot{l}_1+\dot{l}_2+\dot{l})!(\dot{l}_1-\dot{m}_1)!(\dot{l}_2-\dot{m}_2)!}\right)^{\frac{1}{2}}\times\\
{}_3F_2\left(\begin{matrix}l+m+1,-l+m,-l_1+m_1\\ -l_1-l_2+m,l_2-l_1+m+1\end{matrix}\middle|\,1\right){}_3F_2\left(\begin{matrix}\dot{l}+\dot{m}+1,-\dot{l}+\dot{m},-\dot{l}_1+\dot{m}_1\\ -\dot{l}_1-\dot{l}_2+\dot{m},\dot{l}_2-\dot{l}_1+\dot{m}+1\end{matrix}\middle|\,1\right),
\end{gathered}\tag{125}$$

where $m=m_1+m_2$, $\dot{m}=\dot{m}_1+\dot{m}_2$. In virtue of the orthogonality of the Clebsch–Gordan coefficients from (124) it follows that

$$\begin{aligned}
\zeta_{l_1m_1;\dot{l}_1\dot{m}_2}\otimes\zeta_{l_2m_2;\dot{l}_2\dot{m}_2} &= \sum_{\substack{m_1+m_2=m,\\ \dot{m}_1+\dot{m}_2=\dot{m}}}\overline{C(l_1,l_2,l;m_1,m_2,m)C(\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m})}\zeta_{lm;\dot{l}\dot{m}}\\
&= \sum_{\substack{m_1+m_2=m,\\ \dot{m}_1+\dot{m}_2=\dot{m}}}\bar{B}^{l_1,l_2,l;m_1,m_2,m}_{\dot{l}_1,\dot{l}_2,\dot{l};\dot{m}_1,\dot{m}_2,\dot{m}}\zeta_{lm;\dot{l}\dot{m}}.
\end{aligned}\tag{126}$$

Further, assume that $l_1=1$ and $\dot{l}_1=1$, then at $l_2\geq 1$ and $\dot{l}_2\geq 1$ the numbers l and m (correspondingly $\dot{l}$ and $\dot{m}$) take the values $l=l_2-1$, l_2, l_2+1, $m=m_2-1$, m_2, m_2+1 (correspondingly $\dot{l}=\dot{l}_2-1$, $\dot{l}_2$, $\dot{l}_2+1$, $\dot{m}=\dot{m}_2-1$, $\dot{m}_2$, $\dot{m}_2+1$). In this case the system (126) can be rewritten as follows

$$\begin{aligned}
\zeta_{1,-1;1,-1}\otimes\zeta_{l_2,m_2+1;\dot{l}_2,\dot{m}_2+1} &= \bar{b}^m_{11}\zeta_{l+1,m;\dot{l}+1,\dot{m}}+\bar{b}^m_{12}\zeta_{lm;\dot{l}\dot{m}}+\bar{b}^m_{13}\zeta_{l-1,m;\dot{l}-1,\dot{m}},\\
\zeta_{1,0;1,0}\otimes\zeta_{l_2m_2;\dot{l}_2\dot{m}_2} &= \bar{b}^m_{21}\zeta_{l+1,m;\dot{l}+1,\dot{m}}+\bar{b}^m_{22}\zeta_{lm;\dot{l}\dot{m}}+\bar{b}^m_{23}\zeta_{l-1,m;\dot{l}-1,\dot{m}},\\
\zeta_{1,1;1,1}\otimes\zeta_{l_2,m_2-1;\dot{l}_2,\dot{m}_2-1} &= \bar{b}^m_{31}\zeta_{l+1,m;\dot{l}+1,\dot{m}}+\bar{b}^m_{32}\zeta_{lml\dot{m}}+\bar{b}^m_{33}\zeta_{l-1,m;\dot{l}-1,\dot{m}},
\end{aligned}\tag{127}$$

where

$$\bar{b}^{(m)}=\begin{pmatrix}
\bar{B}^{1,l_2,l_2-1;1,m_2,m}_{1,\dot{l}_2,\dot{l}_2-1;1,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2-1;0,m_2,m}_{1,\dot{l}_2,\dot{l}_2-1;1,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2-1;-1,m_2,m}_{1,\dot{l}_2,\dot{l}_2-1;-1,\dot{m}_2,\dot{m}}\\
\bar{B}^{1,l_2,l_2;1,m_2,m}_{1,\dot{l}_2,\dot{l}_2;1,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2;0,m_2,m}_{1,\dot{l}_2,\dot{l}_2;0,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2;-1,m_2,m}_{1,\dot{l}_2,\dot{l}_2;-1,\dot{m}_2,\dot{m}}\\
\bar{B}^{1,l_2,l_2+1;1,m_2,m}_{1,\dot{l}_2,\dot{l}_2+1;1,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2+1;0,m_2,m}_{1,\dot{l}_2,\dot{l}_2+1;0,\dot{m}_2,\dot{m}} & \bar{B}^{1,l_2,l_2+1;-1,m_2,m}_{1,\dot{l}_2,\dot{l}_2+1;-1,\dot{m}_2,\dot{m}}
\end{pmatrix}=$$

$$
\begin{pmatrix}
\sqrt{\frac{(l_2-m)(l_2-m+1)(\dot{l}_2-\dot{m})(\dot{l}_2-\dot{m}+1)}{(2l_2+1)(2l_2+2)(2\dot{l}_2+1)(2\dot{l}_2+2)}} & \sqrt{\frac{(l_2+m+1)(l_2-m)(\dot{l}_2+\dot{m}+1)(\dot{l}_2-\dot{m}}{4l_2(l_2+1)\dot{l}_2(\dot{l}_2+1)}} & \sqrt{\frac{(l_2+m)(l_2+m+1)(\dot{l}_2+\dot{m})(\dot{l}_2+\dot{m}+1)}{4l_2(2l_2+1)\dot{l}_2(\dot{l}_2+1)}} \\
\sqrt{\frac{(l_2+m+1)(l_2-m+1)(\dot{l}_2+\dot{m}+1)(\dot{l}_2-\dot{m}+1)}{(2l_2+1)(l_2+1)(2\dot{l}_2+1)(\dot{l}_2+1)}} & \sqrt{\frac{m\dot{m}}{l_2(l_2+1)\dot{l}_2(\dot{l}_2+1)}} & \sqrt{\frac{(l_2+m_2)(l_2-m_2)(\dot{l}_2+\dot{m}_2)(\dot{l}_2-\dot{m}_2)}{l_2(2l_2+1)\dot{l}_2(2\dot{l}_2+1)}} \\
\sqrt{\frac{(l_2+m)(l_2+m+1)(\dot{l}_2+\dot{m})(\dot{l}_2+\dot{m}+1)}{(2l_2+1)(2l_2+2)(2\dot{l}_2+1)(2\dot{l}_2+2)}} & \sqrt{\frac{(l_2+m)(l_2-m+1)(\dot{l}_2+\dot{m})(\dot{l}_2-\dot{m}+1)}{4l_2(l_2+1)\dot{l}_2(\dot{l}_2+1)}} & \sqrt{\frac{(l_2-m)(l_2-m+1)(\dot{l}_2-\dot{m})(\dot{l}_2-\dot{m}+1)}{4l_2(2l_2+1)\dot{l}_2(\dot{l}_2+1)}}
\end{pmatrix}. \quad (128)
$$

Let $T^l_{\mathfrak{g}}$ be a matrix of the irreducible representation of the weight l in the helicity basis. Let us apply the transformation $T_{\mathfrak{g}}$ to the left and right parts of the each equalities (127). In the left part we have

$$
T_{\mathfrak{g}}\zeta_{1,k;1,\dot{k}} \otimes \zeta_{l_2,m_2-k;\dot{l}_2-\dot{k}} = T^1_{\mathfrak{g}}\zeta_{1,k;1,\dot{k}} \otimes T^{l_2}_{\mathfrak{g}}\zeta_{l_2,m_2-k;\dot{l}_2-\dot{k}} =
\bar{b}^m_{k+2,1}T^{l+1}_{\mathfrak{g}}\zeta_{l+1,m;\dot{l}+1} + \bar{b}^m_{k+2,2}T^l_{\mathfrak{g}}\zeta_{lm;\dot{l}\dot{m}} + \bar{b}^m_{k+2,3}T^{l-1}_{\mathfrak{g}}\zeta_{l-1,m;\dot{l}-1,\dot{m}},
$$

where $k, \dot{k} = -1, 0, 1$. Denoting the elements of $T^l_{\mathfrak{g}}$ via $\mathfrak{M}^l_{mn}$ (generalized hyperspherical functions), we find

$$
\left(\mathfrak{M}^1_{-1,k}\zeta_{1,-1;1,-1} + \mathfrak{M}^1_{0,k}\zeta_{1,0;1,0} + \mathfrak{M}^1_{1,k}\zeta_{1,1;1,1}\right)\sum \mathfrak{M}^l_{j,m-k}\zeta_{l_2,m_2-j;\dot{l}_2-\dot{j}} =
\sum\left(\bar{b}^m_{k+2,1}\mathfrak{M}^{l+1}_{jm}\zeta_{l+1,j;\dot{l}+1,\dot{j}} + \bar{b}^m_{k+2,2}\mathfrak{M}^l_{jm}\zeta_{lj;\dot{l}\dot{j}} + \bar{b}^m_{k+2,3}\mathfrak{M}^{l-1}_{jm}\zeta_{l-1,j;\dot{l}-1;\dot{j}}\right).
$$

Replacing in the right part the vectors $\zeta_{l+1,j;\dot{l}+1,\dot{j}}$, $\zeta_{lj;\dot{l}\dot{j}}$, $\zeta_{l-1,j;\dot{l}-1,\dot{j}}$ via $\zeta_{1,-1;1,-1} \otimes \zeta_{l_2,j+1;\dot{l}_2,\dot{j}+1}$, $\zeta_{1,0;1,0}\otimes\zeta_{l_2j;\dot{l}_2\dot{j}}$, $\zeta_{1,1;1,1}\otimes\zeta_{l_2,j-1;\dot{l}_2,\dot{j}-1}$ and comparising the coefficients at $\zeta_{1,-1;1,-1}\otimes\zeta_{l_2,j+1;\dot{l}_2,\dot{j}+1}$, $\zeta_{1,0;1,0} \otimes \zeta_{l_2j;\dot{l}_2\dot{j}}$, $\zeta_{1,1;1,1} \otimes \zeta_{l_2,j-1;\dot{l}_2,\dot{j}-1}$ in the left and right parts, we obtain three relations depending on k. Giving in these relations three possible values $-1, 0, 1$ to the number k and substituting the functions $\mathfrak{M}^1_{mn}(\varphi, \epsilon, \theta, \tau, \psi, \varepsilon)$ (the matrix (66)), we find the following nine

recurrence relations:

$$\bar{b}^m_{11}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{11}+\bar{b}^m_{12}\mathfrak{M}^l_{jm}\bar{b}^j_{12}+\bar{b}^m_{13}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{13}=$$
$$\left(\cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}\right)e^{\epsilon+i\varphi+\varepsilon+i\psi}\mathfrak{M}^l_{j+1,m+1},$$
$$\bar{b}^m_{11}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{21}+\bar{b}^m_{12}\mathfrak{M}^l_{jm}\bar{b}^j_{22}+\bar{b}^m_{13}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{23}=$$
$$\frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau)e^{-\varepsilon-i\psi}\mathfrak{M}^l_{j,m+1},$$
$$\bar{b}^m_{11}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{31}+\bar{b}^m_{12}\mathfrak{M}^l_{jm}\bar{b}^j_{32}+\bar{b}^m_{13}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{33}=$$
$$\left(\cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}\right)e^{\epsilon+i\varphi-\varepsilon-i\psi}\mathfrak{M}^l_{j-1,m+1},$$
$$\bar{b}^m_{21}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{11}+\bar{b}^m_{22}\mathfrak{M}^l_{jm}\bar{b}^j_{12}+\bar{b}^m_{23}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{13}=$$
$$\frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau)e^{\varepsilon+i\psi}\mathfrak{M}^l_{j+1,m},$$
$$\bar{b}^m_{21}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{21}+\bar{b}^m_{22}\mathfrak{M}^l_{jm}\bar{b}^j_{22}+\bar{b}^m_{23}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{23}=(\cos\theta\cosh\tau+i\sin\theta\sinh\tau)\mathfrak{M}^l_{j,m},$$
$$\bar{b}^m_{21}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{31}+\bar{b}^m_{22}\mathfrak{M}^l_{jm}\bar{b}^j_{32}+\bar{b}^m_{23}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{33}=$$
$$\frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau)e^{-\varepsilon-i\psi}\mathfrak{M}^l_{j-1,m},$$
$$\bar{b}^m_{31}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{11}+\bar{b}^m_{32}\mathfrak{M}^l_{jm}\bar{b}^j_{12}+\bar{b}^m_{33}\mathfrak{M}^{l-1}_{jm}\bar{b}^i_{13}=$$
$$\left(\cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}\right)e^{-\epsilon-i\varphi+\epsilon+i\psi}\mathfrak{M}^l_{j+1,m-1},$$
$$\bar{b}^m_{31}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{21}+\bar{b}^m_{32}\mathfrak{M}^l_{jm}\bar{b}^j_{22}+\bar{b}^m_{33}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{23}=$$
$$\frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau)e^{-\epsilon-i\varphi}\mathfrak{M}^l_{j,m-1},$$
$$\bar{b}^m_{31}\mathfrak{M}^{l+1}_{jm}\bar{b}^j_{31}+\bar{b}^m_{32}\mathfrak{M}^l_{jm}\bar{b}^j_{32}+\bar{b}^m_{33}\mathfrak{M}^{l-1}_{jm}\bar{b}^j_{33}=$$
$$\left(\cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}\right)e^{-\epsilon-i\varphi-\varepsilon-i\psi}\mathfrak{M}^l_{j-1,m-1}.$$

Let us find recurrence relations between the functions $\mathfrak{M}^l_{mn}$, where the weight l changed by $\frac{1}{2}$. Thus, at $l_1=1/2$, $\dot{l}_2=1/2$ and $l_1\geq 1/2$, $\dot{l}_2\geq 1/2$ by analogy with (127) we obtain

$$\begin{aligned}
\zeta_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}}\otimes\zeta_{l_2,m_2-\frac{1}{2};\dot{l}_2,\dot{m}_2-\frac{1}{2}} &= \bar{b}^m_{00}\zeta_{l+\frac{1}{2},m;\dot{l}+\frac{1}{2},\dot{m}}+\bar{b}^m_{01}\zeta_{l-\frac{1}{2},m;\dot{l}-\frac{1}{2},\dot{m}},\\
\zeta_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}}\otimes\zeta_{l_2,m_2+\frac{1}{2};\dot{l}_2,\dot{m}_2+\frac{1}{2}} &= \bar{b}^m_{10}\zeta_{l+\frac{1}{2},m;\dot{l}+\frac{1}{2},\dot{m}}+\bar{b}^m_{11}\zeta_{l-\frac{1}{2},m;\dot{l}-\frac{1}{2},\dot{m}},
\end{aligned}$$

where

$$\bar{b}^{(m)}=\begin{pmatrix}\bar{B}^{\frac{1}{2},l_2,l_2+\frac{1}{2};-\frac{1}{2},m_2,m}_{\frac{1}{2},\dot{l}_2,\dot{l}_2+\frac{1}{2};-\frac{1}{2},\dot{m}_2,\dot{m}} & \bar{B}^{\frac{1}{2},l_2,l_2-\frac{1}{2};-\frac{1}{2},m_2,m}_{\frac{1}{2},\dot{l}_2,\dot{l}_2-\frac{1}{2};-\frac{1}{2},\dot{m}_2,\dot{m}}\\ \bar{B}^{\frac{1}{2},l_2,l_2+\frac{1}{2};\frac{1}{2},m_2,m}_{\frac{1}{2},\dot{l}_2,\dot{l}_2+\frac{1}{2};\frac{1}{2},\dot{m}_2,\dot{m}} & \bar{B}^{\frac{1}{2},l_2,l_2-\frac{1}{2};\frac{1}{2},m_2,m}_{\frac{1}{2},\dot{l}_2,\dot{l}_2-\frac{1}{2};\frac{1}{2},\dot{m}_2,\dot{m}}\end{pmatrix}=$$
$$\begin{pmatrix}\sqrt{\frac{(l_2-m+\frac{1}{2})(\dot{l}_2-\dot{m}+\frac{1}{2})}{(2l_2+1)(2\dot{l}_2+1)}} & \sqrt{\frac{(l_2+m+\frac{1}{2})(\dot{l}_2+\dot{m}+\frac{1}{2})}{(2l_2+1)(2\dot{l}_2+1)}}\\ \sqrt{\frac{(l_2+m+\frac{1}{2})(\dot{l}_2+\dot{m}+\frac{1}{2})}{(2l_2+1)(2\dot{l}_2+1)}} & \sqrt{\frac{(l_2-m+\frac{1}{2})(\dot{l}_2-\dot{m}+\frac{1}{2})}{(2l_2+1)(2\dot{l}_2+1)}}\end{pmatrix}.$$

Carrying out the analogous calculations as for the case $l=1$ and using the matrix (65), we come to the following recurrence relations

$$
\begin{aligned}
\bar{b}^m_{00}\mathfrak{M}^{l+\frac{1}{2}}_{jm}\bar{b}^j_{00}+\bar{b}^m_{01}\mathfrak{M}^{l-\frac{1}{2}}_{jm}\bar{b}^j_{01} &= \left(\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right)e^{\frac{\epsilon+i\varphi+\varepsilon+i\psi}{2}}\mathfrak{M}^l_{j+\frac{1}{2},m+\frac{1}{2}},\\
\bar{b}^m_{00}\mathfrak{M}^{l+\frac{1}{2}}_{jm}\bar{b}^j_{10}+\bar{b}^m_{01}\mathfrak{M}^{l-\frac{1}{2}}_{jm}\bar{b}^j_{11} &= \left(\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right)e^{\frac{\epsilon+i\varphi-\varepsilon-i\psi}{2}}\mathfrak{M}^l_{j-\frac{1}{2},m+\frac{1}{2}},\\
\bar{b}^m_{10}\mathfrak{M}^{l+\frac{1}{2}}_{jm}\bar{b}^j_{00}+\bar{b}^m_{11}\mathfrak{M}^{l-\frac{1}{2}}_{jm}\bar{b}^j_{01} &= \left(\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right)e^{\frac{-\epsilon-i\varphi+\varepsilon+i\psi}{2}}\mathfrak{M}^l_{j+\frac{1}{2},m-\frac{1}{2}},\\
\bar{b}^m_{10}\mathfrak{M}^{l+\frac{1}{2}}_{jm}\bar{b}^j_{10}+\bar{b}^m_{11}\mathfrak{M}^{l-\frac{1}{2}}_{jm}\bar{b}^j_{11} &= \left(\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right)e^{\frac{-\epsilon-i\varphi-\varepsilon-i\psi}{2}}\mathfrak{M}^l_{j-\frac{1}{2},m-\frac{1}{2}}.
\end{aligned}
$$

6 Harmonic analysis on the group $SU(2)\otimes SU(2)$

Since $SU(2)\otimes SU(2)$ is locally compact, then there exists an invariant measure $d\mathfrak{g}$ (Haar measure) on this group, that is, such a measure that for any finite continuous function $f(\mathfrak{g})$ we have

$$\int f(\mathfrak{g})d\mathfrak{g}=\int f(\mathfrak{g}_0\mathfrak{g})d\mathfrak{g}=\int f(\mathfrak{g}\mathfrak{g}_0)d\mathfrak{g}=\int f(\mathfrak{g}^{-1})d\mathfrak{g}.$$

Applying the equations (95)–(100), we come to a following expression for the Haar measure in terms of the parameters (44):

$$d\mathfrak{g}=\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon. \tag{129}$$

Thus, an invariant integration on the group $SU(2)\otimes SU(2)$ is defined by the formula

$$\int\limits_{SU(2)\otimes SU(2)} f(g)d\mathfrak{g}=\frac{1}{32\pi^4}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} f(\theta,\varphi,\psi,\tau,\epsilon,\varepsilon)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

Since a dimension of the spinor representation $T_{\mathfrak{g}}$ of $SU(2)\otimes SU(2)$ is equal to $(2l+1)(2\dot{l}+1)$, then the functions $\sqrt{(2l+1)(2\dot{l}+1)}t^l_{mn}(\mathfrak{g})$ form a full orthogonal normalized system on this group with respect to the invariant measure $d\mathfrak{g}$. At this point, the index l runs all possible integer or half-integer non-negative values, and the indices m and n run the values $-l,-l+1,\dots,l-1,l$. In virtue of (47) the matrix elements t^l_{mn} are expressed via the generalized hyperspherical function $t^l_{mn}(\mathfrak{g})=\mathfrak{M}^l_{mn}(\varphi,\epsilon,\theta,\tau,\psi,\varepsilon)$. Therefore,

$$\int\limits_{SU(2)\otimes SU(2)}\mathfrak{M}^l_{mn}(\mathfrak{g})\overline{\mathfrak{M}^l_{mn}(\mathfrak{g})}d\mathfrak{g}=\frac{32\pi^4}{(2l+1)(2\dot{l}+1)}\delta(\mathfrak{g}'-\mathfrak{g}), \tag{130}$$

where $\delta(\mathfrak{g}'-\mathfrak{g})$ is a δ-function on the group $SU(2)\otimes SU(2)$. An explicit form of δ-function is

$$
\begin{aligned}
\delta(\mathfrak{g}'-\mathfrak{g})=\delta(\varphi'-\varphi)\delta(\epsilon'-\epsilon)\delta(\cos\theta'\cosh\tau'-\cos\theta\cosh\tau)\times\\
\times\,\delta(\sin\theta'\sinh\tau'-\sin\theta\sinh\tau)\delta(\psi'-\psi)\delta(\varepsilon'-\varepsilon).
\end{aligned}
$$

Substituting into (130) the expression

$$\mathfrak{M}^l_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)} Z^l_{mn} e^{-n(\varepsilon+i\psi)}$$

and taking into account (129), we obtain

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} Z^l_{mn}\overline{Z^s_{pq}} e^{-(m+p)\epsilon} e^{-i(m-p)\varphi}\times$$
$$\times\, e^{-(n+q)\varepsilon} e^{-i(n-q)\psi} \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon = \frac{32\pi^4\delta_{ls}\delta_{mp}\delta_{nq}\delta(\mathfrak{g}'-\mathfrak{g})}{(2l+1)(2\dot{l}+1)}.$$

Since the hyperspherical function of the principal series has the form

$$Z^{-\frac{1}{2}+i\rho}_{mn}(\cos\theta^c) = \sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]} P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau),$$

then any function $f(\mathfrak{g})$ on the group $SU(2)\otimes SU(2)$, such that $\int|f(\mathfrak{g})|^2 d\mathfrak{g} < +\infty$, is expanded on the matrix elements of the principal series

$$\mathfrak{M}^{-\frac{1}{2}+i\rho}_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)} \sum_{k=-l}^{\left[\frac{\lambda}{2}\right]} P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau)$$

Since the Lorentz group is noncompact, then an expansion should be run on the conical functions $\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau)$, where the parameter τ changes in the limits $0 \le \tau < \infty$. In other words, this problem can be formulated as follows.

Let m and n be simultaneously integer or half-integer numbers, and let $F(x)$ be the function such that

$$\int\limits_{1}^{\infty} |F(x)|^2 dx < +\infty.$$

It takes to expand the function $F(x)$ via the functions $\mathfrak{P}^l_{mn}(x)$, where l is an expansion parameter and $1 \le x < \infty$.

As it shown in [76], the functions $\mathfrak{P}^l_{mn}(x)$ satisfy the self-conjugate differential equation

$$\frac{d}{dx}(x^2-1)\frac{du}{dx} - \frac{m^2+n^2-2mnx}{x^2-1} = l(l+1)u.$$

The functions $\mathfrak{P}^l_{mn}(x)$ are continuous at the point $x=1$ corresponding to $\tau=0$. In other words, $\mathfrak{P}^l_{mn}(x)$ are eigenfunctions of the self-conjugate operator

$$\frac{d}{dx}(x^2-1)\frac{du}{dx} - \frac{m^2+n^2-2mnx}{x^2-1}, \quad 1 \le x < \infty.$$

Using the standard expansion technique on eigenfunctions of self-conjugate operators (see, for example, [43, 46]), Vilenkin [76] derived the following result.

Theorem 2 (Vilenkin [76]). *If m and n are integer numbers, then any function $F(x)$, such that $\int\limits_1^\infty |F(x)|^2 dx < +\infty$, has an expansion*

$$F(x) = \frac{1}{4\pi^2}\int\limits_0^\infty a(\rho)\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(x)\rho\tanh\pi\rho d\rho + \frac{1}{4\pi^2}\sum_{l=1}^{M}\left(l-\frac{1}{2}\right)b(l)\mathfrak{P}^l_{mn}(x),$$

where

$$M = \begin{cases} \min(|m|,|n|) & \text{if } mn \geq 0,\\ 0 & \text{if } mn < 0. \end{cases}$$

The summation is produced via integer values l. The coefficients $a(\rho)$ and $b(l)$ in this expansion are expressed by the formulas:

$$a(\rho) = \int\limits_1^\infty F(x)\mathfrak{P}_{mn}^{-\frac{1}{2}-i\rho}(x)dx \equiv \int\limits_1^\infty F(x)\overline{\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(x)}dx$$

and

$$b(l) = (-1)^{m-n}\int\limits_1^\infty F(x)\mathfrak{P}_{mn}^{-l-1}(x)dx \equiv$$

$$\equiv (-1)^{m-n}\frac{\Gamma(l+m+1)\Gamma(l-m+1)}{\Gamma(l+n+1)\Gamma(l-n+1)}\int\limits_1^\infty F(x)\overline{\mathfrak{P}^l_{mn}(x)}dx.$$

There is an analog of the Plancherel formula:

$$\int\limits_1^\infty |F(x)|^2dx = \frac{1}{4\pi^2}\int\limits_0^\infty |a(\rho)|^2\rho\tanh\pi\rho d\rho +$$

$$+\frac{(-1)^{n-m}}{4\pi^2}\sum_{l=1}^{M}\left(l-\frac{1}{2}\right)\frac{\Gamma(l+n+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l-m+1)}|b(l)|^2.$$

Analogously, if m and n are half-integer numbers, then

$$F(x) = \frac{1}{4\pi^2}\int\limits_0^\infty a(\rho)\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(x)\rho\coth\pi\rho d\rho + \frac{1}{4\pi^2}\sum_{l=\frac{1}{2}}^{M}\left(l-\frac{1}{2}\right)b(l)\mathfrak{P}^l_{mn}(x),$$

where

$$M = \begin{cases} \min(|m|,|n|) & \text{if } mn > 0,\\ 0 & \text{if } mn < 0. \end{cases}$$

The summation is produced via half-integer values l. The coefficients $a(\rho)$ and $b(l)$ are expressed by the same formulas that take place for integer m and n. In this case, the Plancherel

formula has a following form

$$\int\limits_{1}^{\infty}|F(x)|^2dx=\frac{1}{4\pi^2}\int\limits_{0}^{\infty}|a(\rho)|^2\rho\coth\pi\rho d\rho+$$
$$+\frac{(-1)^{m-n}}{4\pi^2}\sum^{M}_{l=\frac{1}{2}}\left(l-\frac{1}{2}\right)\frac{\Gamma(l+n+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l-m+1)}|b(l)|^2.$$

Let us apply the Vilenkin Theorem to an expansion of the functions $f(\mathfrak{g})$ on the Lorentz group, such that $\int|f(\mathfrak{g})|^2d\mathfrak{g}<+\infty$. Then, taking into account that matrix elements of the principal series representations have the form

$$\mathfrak{M}^{-\frac{1}{2}+i\rho}_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\sum^{[\frac{\lambda}{2}]}_{k=-\frac{\lambda}{2}}P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau),$$

we obtain

$$f(\mathfrak{g})=\frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4}\sum_{m,n}\sum^{[\frac{\lambda}{2}]}_{k=-\frac{\lambda}{2}}\left[\int\limits_{0}^{\infty}a_{mn}(\rho)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times\right.$$
$$\times P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau)\rho\tanh\pi\rho d\rho+$$
$$\left.+\sum^{\infty}_{l_0=1}\left(l_0-\frac{1}{2}\right)b_{mn}(l_0)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{l_0}_{mk}(\cos\theta)\mathfrak{P}^{l_0}_{kn}(\cosh\tau)\right]. \quad (131)$$

The values of the coefficients $a_{mn}(\rho)$ and $b_{mn}(l_0)$ are expressed by the formulas

$$a_{mn}(\rho)=\sum^{[\frac{\lambda}{2}]}_{k=-\frac{\lambda}{2}}\int f(\mathfrak{g})e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}-i\rho}_{kn}(\cosh\tau)d\mathfrak{g}, \quad (132)$$

$$b_{mn}(l_0)=\frac{(-1)^{m-n}\Gamma(l_0+m+1)\Gamma(l_0-m+1)}{\Gamma(l_0+n+1)\Gamma(l_0-n+1)}\int f(\mathfrak{g})\mathfrak{M}^{l_0}_{mn}(\mathfrak{g})d\mathfrak{g}, \quad (133)$$

where $d\mathfrak{g}$ is the Haar measure on the Lorentz group in Euler parameters:

$$d\mathfrak{g}=\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

In the case of discrete series of representations, the expansion takes a form

$$f(\mathfrak{g})=\frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4}\sum_{m,n}\sum^{[\frac{\lambda}{2}]}_{k=-\frac{\lambda}{2}}\left[\int\limits_{0}^{\infty}a_{mn}(\rho)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times\right.$$
$$\times P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau)\rho\tanh\pi\rho d\rho+$$
$$\left.+\sum^{M}_{l_0=1}\left(l_0-\frac{1}{2}\right)b_{mn}(l_0)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{l_0}_{mk}(\cos\theta)\mathfrak{P}^{l_0}_{kn}(\cosh\tau)\right]. \quad (134)$$

where

$$M=\begin{cases}\min(|m|,|n|) & \text{if } mn\geq 0,\\ 0 & \text{if } mn<0.\end{cases}$$

The coefficients $a_{mn}(\rho)$ and $b_{mn}(l)$ have the form (132) and (133). Analogously, if m and n are half-integer numbers, then

$$f(\mathfrak{g})=\frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4}\sum_{m,n}\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}\left[\int\limits_0^\infty a_{mn}(\rho)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times\right.$$
$$\times P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{-\frac{1}{2}+i\rho}_{kn}(\cosh\tau)\rho\coth\pi\rho d\rho+$$
$$\left.+\sum_{l_0=\frac{1}{2}}^{M}\left(l_0-\frac{1}{2}\right)b_{mn}(l_0)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{l_0}_{mk}(\cos\theta)\mathfrak{P}^{l_0}_{kn}(\cosh\tau)\right]. \quad (135)$$

where

$$M=\begin{cases}\min(|m|,|n|) & \text{if } mn>0,\\ 0 & \text{if } mn<0.\end{cases}$$

The coefficients $a_{mn}(\rho)$ and $b_{mn}(l_0)$ have the same form as in the case of integer m and n.

Since

$$\coth\pi\rho=\tanh\pi\left(\rho+\frac{i}{2}\right),$$

then integral terms in the expansions (134) and (135) can be written uniformly if we replace $\tanh\pi\rho$ and $\coth\pi\rho$ by $\tanh\pi(\rho+oi)$, where $o=0$ in the integer case and $o=1/2$ in the half-integer case. The following expansion is an unification of the expansions (134) and (135):

$$f(\mathfrak{g})=\frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4}\sum_{m,n}\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}\left[\int\limits_0^\infty a^o_{mn}(\rho)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times\right.$$
$$\times P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{(-\frac{1}{2}+i\rho,o)}_{kn}(\cosh\tau)\rho\tanh\pi(\rho+oi)d\rho+$$
$$\left.+\sum_{l_0=1-o}^{M}\left(l_0-\frac{1}{2}\right)b^o_{mn}(l_0)e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{l_0}_{mk}(\cos\theta)\mathfrak{P}^{l_0}_{kn}(\cosh\tau)\right], \quad (136)$$

where

$$a^o_{mn}(\rho)=\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}\int f(\mathfrak{g})e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}P^{\frac{\lambda}{2}}_{mk}(\cos\theta)\mathfrak{P}^{(-\frac{1}{2}+i\rho,o)}_{kn}(\cosh\tau)d\mathfrak{g}, \quad (137)$$

$$b^o_{mn}(l_0)=\frac{(-1)^{m-n}\Gamma(l_0+m+o+1)\Gamma(l_0-m-o+1)}{\Gamma(l_0+n+o+1)\Gamma(l_0-n-o+1)}\int f(\mathfrak{g})\mathfrak{M}^{l_0}_{mn}(\mathfrak{g})d\mathfrak{g}, \quad (138)$$

$$d\mathfrak{g}=\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

There is an analog of the Plancherel formula:

$$\int |f(\mathfrak{g})|^2 d\mathfrak{g} = \frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4} \sum_{m,n,o} \left[\int\limits_0^\infty |a^o_{mn}(\rho)|^2 \rho \tanh \pi(\rho + oi) d\rho + \right.$$

$$\left. + (-1)^{m-n} \sum_{l_0=1-o}^{M} \frac{\Gamma(l_0+n+o+1)\Gamma(l_0-n-o+1)}{\Gamma(l_0+m+o+1)\Gamma(l_0-m-o+1)} \left(l_0 - \frac{1}{2}\right) |b^o_{mn}(l_0)|^2 \right].$$

Thus, the expansion of square integrable functions $f(\mathfrak{g})$ on the group $SL(2,\mathbb{C})$ has only matrix elements of the principal series of representations. Representations of supplementary series do not participate in the expansion.

An expansion of the functions $f(\mathfrak{g})$ on the Lorentz group in terms of associated hyperspherical functions, that is, an expansion on the two-dimensional complex sphere, has an important meaning in physical applications. Using the explicit expression (76) for the associated hyperspherical functions of the principal series, we find the following expansion:

$$f(\mathfrak{g}) = \frac{(2l_0+1)(2\dot{l}_0+1)}{32\pi^4} \sum_m \sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]} \left[\int\limits_0^\infty a^o_m(\rho) e^{-m(\epsilon+i\varphi)} \times \right.$$

$$\times P^{\frac{\lambda}{2}}_{mk}(\cos\theta) \mathfrak{P}^k_{\left(-\frac{1}{2}+i\rho,o\right)}(\cosh\tau) \rho \tanh \pi(\rho+oi) d\rho +$$

$$\left. + \sum_{l_0=1-o}^{M} \left(l_0 - \frac{1}{2}\right) b^o_m(l_0) e^{-m(\epsilon+i\varphi)} P^{l_0}_{mk}(\cos\theta) \mathfrak{P}^k_{l_0}(\cosh\tau) \right], \quad (139)$$

where

$$a^o_m(\rho) = \sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]} \int f(\mathfrak{g}) e^{-m(\epsilon+i\varphi)} P^{\frac{\lambda}{2}}_{mk}(\cos\theta) \mathfrak{P}^k_{\left(-\frac{1}{2}+i\rho,o\right)}(\cosh\tau) d\mathfrak{g}, \quad (140)$$

$$b^o_m(l_0) = \frac{(-1)^m \Gamma(l_0+m+o+1)\Gamma(l_0-m-o+1)}{\Gamma(l_0+o+1)\Gamma(l_0-o+1)} \int f(\mathfrak{g}) \mathfrak{M}^m_{l_0}(\mathfrak{g}) d\mathfrak{g} \quad (141)$$

and

$$d\mathfrak{g} = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon.$$

is a Haar measure on the surface of the two-dimensional complex sphere.

In turn, physical fields, describing particles with an arbitrary spin (both integer and half-integer), are defined in terms of finite-dimensional (spinor) representations of the group $\mathfrak{G}_+$ via the expansions on associated hyperspherical functions realized on the surface of the two-dimensional complex sphere. At this point, the expansion (139) takes a form

$$f(\mathfrak{g}) = \frac{(2l+1)(2\dot{l}+1)}{32\pi^4} \sum_{k=-l}^{l} \sum_{l=1-o}^{M} \left(l - \frac{1}{2}\right) b^o_m(l) e^{-m(\epsilon+i\varphi)} P^l_{mk}(\cos\theta) \mathfrak{P}^k_l(\cosh\tau), \quad (142)$$

where

$$b^o_m(l) = \frac{(-1)^m \Gamma(l+m+o+1)\Gamma(l-m-o+1)}{\Gamma(l+o+1)\Gamma(l-o+1)} \int f(\mathfrak{g}) \mathfrak{M}^m_l(\mathfrak{g}) d\mathfrak{g}. \quad (143)$$

An expansion on the cone for the square integrable function (the wave function of free particle with the null spin) was first given by I. S. Shapiro [58, 59]. Later on, Chou Kuang-chao and Zastavenko [13] refined the Shapiro expansion for the particles with non-null spin (see also [15, 49]). From mathematical viewpoint the functions obtained in [58, 13] present an integral transformation, which allows one to obtain an expansion of the function, defined on the hyperboloid, via the conical functions. Such expansions with the use of integral geometry have been further studied in the Gel'fand-Graev works [24] and later have been used by Vilenkin and Smorodinsky [75] for a definition of the formulas of direct and inverse expansions on degenerate representations of the Lorentz group in different coordinate systems. Further, in sequel of the work [75] the authors of [81, 41] studied the questions about relation between different subgroups of the Lorentz group and coordinate systems on the hyperboloid and also the questions concerning convergence and asymptotic expansions. The following step in this direction was became in [44, 42], where the authors constructed irreducible unitary representations of the Lorentz group, realized on the space of functions defined on the direct product of two spaces – the hyperboloid (with the infinite-dimensional representation) and the sphere (with the finite-dimensional spinor representation). These functions, forming the basis for integral representations of scattering amplitudes, are represented by the product of spherical functions (Legendre functions) and conical functions (see also[48]). In the work [42] it has been shown that a realization of the Lorentz group representations on the cone is closely related with the formulas of helicity expansions [37]. Basis functions on the cone and relativistically-invariant expansions of the spiral scattering amplitudes have been obtained by Verdiev [73, 72].

Example. By way of example let us consider a decomposition of the functions on the group $SU(2)\otimes SU(2)$, defining solutions for the Dirac field $(1/2,0)\oplus(0,1/2)$. In [68, 70] it has been shown that these solutions are defined in terms of associated hyperspherical functions:

$$\begin{aligned}
\psi_1(r,\varphi^c,\theta^c) &= \boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{\frac{1}{2}}_l(\varphi,\epsilon,\theta,\tau,0,0),\\
\psi_2(r,\varphi^c,\theta^c) &= \pm\boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{-\frac{1}{2}}_l(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}_1(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \mp\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\frac{1}{2}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}_2(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{-\frac{1}{2}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0),
\end{aligned}$$

where

$$l=\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots;$$
$$\dot{l}=\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots;$$

$$\mathfrak{M}^{\pm\frac{1}{2}}_l(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon+i\varphi)}Z^{\pm\frac{1}{2}}_l(\theta,\tau),$$

$$Z^{\pm\frac{1}{2}}_l(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^{l}_{k=-l}i^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{array}{c}\pm\frac{1}{2}-l+1,1-l-k\\ \pm\frac{1}{2}-k+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}-l+1,1-l-k\\ -k+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right),$$

$$\mathfrak{M}^{\pm\frac{1}{2}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon-i\varphi)}Z^{\pm\frac{1}{2}}_{\dot{l}}(\theta,\tau),$$

$$Z^{\pm\frac{1}{2}}_{\dot{l}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum^{\dot{l}}_{\dot{k}=-\dot{l}}i^{\pm\frac{1}{2}-\dot{k}}\tan^{\pm\frac{1}{2}-\dot{k}}\frac{\theta}{2}\tanh^{-\dot{k}}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \pm\frac{1}{2}-\dot{k}+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-\dot{l}+1,1-\dot{l}-\dot{k}\\ -\dot{k}+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right).$$

The radial components $\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)$ and $\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)$, where r is the radius of the two-dimensional complex sphere, are expressed via the Bessel functions of half-integer order:

$$\begin{aligned}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r) &= C_1\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\,J_l\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\right)+C_2\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\,J_{-l}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\right),\\ \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*) &= \frac{C_1}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\operatorname{Re} r^*J_{l+1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r^*\right)-\frac{C_2}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\operatorname{Re} r^*J_{-l-1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r^*\right).\end{aligned}$$

Then the decomposition for the field $(1/2,0)\oplus(0,1/2)$ takes a form:

$$\psi_1=\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\sum^{l}_{k=-l}\sum^{M}_{l=\frac{1}{2}}\left(l-\frac{1}{2}\right)b_{\frac{1}{2}}(l)e^{-\frac{1}{2}(\epsilon+i\varphi)}P^{l}_{\frac{1}{2},k}(\cos\theta)\mathfrak{P}^{k}_{l}(\cosh\tau),$$

$$\psi_2=\pm\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\sum^{l}_{k=-l}\sum^{M}_{l=\frac{1}{2}}\left(l-\frac{1}{2}\right)b_{-\frac{1}{2}}(l)e^{\frac{1}{2}(\epsilon+i\varphi)}P^{l}_{-\frac{1}{2},k}(\cos\theta)\mathfrak{P}^{k}_{l}(\cosh\tau),$$

$$\dot{\psi}_1=\mp\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\sum^{\dot{l}}_{\dot{k}=-\dot{l}}\sum^{\dot{M}}_{\dot{l}=\frac{1}{2}}\left(\dot{l}-\frac{1}{2}\right)b_{\frac{1}{2}}(\dot{l})e^{-\frac{1}{2}(\epsilon-i\varphi)}P^{\dot{l}}_{\frac{1}{2},\dot{k}}(\cos\theta)\mathfrak{P}^{\dot{k}}_{\dot{l}}(\cosh\tau),$$

$$\dot{\psi}_2=\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\sum^{\dot{l}}_{\dot{k}=-\dot{l}}\sum^{\dot{M}}_{\dot{l}=\frac{1}{2}}\left(\dot{l}-\frac{1}{2}\right)b_{-\frac{1}{2}}(\dot{l})e^{\frac{1}{2}(\epsilon-i\varphi)}P^{\dot{l}}_{-\frac{1}{2},\dot{k}}(\cos\theta)\mathfrak{P}^{\dot{k}}_{\dot{l}}(\cosh\tau),$$

where

$$b_{\pm\frac{1}{2}}(l)=\frac{(-1)^{\pm\frac{1}{2}}\Gamma(l\pm\frac{1}{2}+1)\Gamma(l\mp\frac{1}{2}+1)}{\Gamma(l+1)\Gamma(l+1)}\int\left[\mathfrak{M}^{\pm\frac{1}{2}}_{l}(\varphi,\epsilon,\theta,\tau,0,0)\right]^2d\mathfrak{g},$$
$$b_{\pm\frac{1}{2}}(\dot{l})=\frac{(-1)^{\pm\frac{1}{2}}\Gamma(\dot{l}\pm\frac{1}{2}+1)\Gamma(\dot{l}\mp\frac{1}{2}+1)}{\Gamma(\dot{l}+1)\Gamma(\dot{l}+1)}\int\left[\mathfrak{M}^{\pm\frac{1}{2}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0)\right]^2d\mathfrak{g},$$

here $d\mathfrak{g}$ is a Haar measure on the surface of the two-dimensional complex sphere:

$$d\mathfrak{g}=\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon.$$

In conclusion, it should be noted that such a description corresponds to a quantum field theory on the Poincaré group introduced by Lurçat [45] (see also [5, 40, 9, 4, 63, 17, 26] and references therein). Moreover, it allows us to widely use a harmonic analysis on the Poincaré group [50, 29, 30] (or, on the product $SU(2)\otimes SU(2)$) at the study of relativistic amplitudes. Harmonic analysis on the Poincaré group in terms of hyperspherical functions will be considered in the following work.

References

[1] A. I. Akhiezer, V. B. Berestetskii, Quantum Electrodynamics (John Wiley & Sons, New York, 1965).

[2] M. Andrews, J. Gunson, *Complex Angular Momenta and Many-Particle States. I. Properties of Local Representations of the Rotation Group*, J. Math. Phys. **5**, 1391–1400 (1964).

[3] P. Appell, M. J. Kampé de Fériet, Fonctions hypégeométriques et hypersphéques. Polynomes d'Hermite (Gauthier–Villars, 1926).

[4] H. Arodź, *Metric tensors, Lagrangian formalism and Abelian gauge field on the Poincaré group*, Acta Phys. Pol., Ser. **B7**, 177–190 (1976).

[5] H. Bacry, A. Kihlberg, *Wavefunctions on homogeneous spaces*, J. Math. Phys. **10**, 2132–2141 (1969).

[6] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48**, 568–640 (1947).

[7] H. Bateman, A. Erdélyi, Higher Transcendental Functions, vol. I (Mc Grow-Hill Book Company, New York, 1953).

[8] H. Bateman, A. Erdélyi, Higher Transcendental Functions, vol. II (Mc Grow-Hill Book Company, New York, 1953).

[9] C. P. Boyer, G. N. Fleming, *Quantum field theory on a seven-dimensional homogeneous space of the Poincaré group*, J. Math. Phys. **15**, 1007–1024 (1974).

[10] J. L. Burchnall, T. W. Chaundy, Quart. J. Math., Oxford Ser. **12**, 112–128 (1941).

[11] E. Cartan, *Sur la détermination d'un systéme orthogonal complet dans un espace dé Riemann symétrique clos*, Rend. Circ. Mat. Palermo **53**, 217–252 (1929).

[12] T. W. Chaundy, *On Appell's fourth hypergeometric function*, Quart. J. Math. **17**, No. 66, 81–85 (1966).

[13] Chou Kuang-chao, I. G. Zastavenko, *The Shapiro integral transformation*, Zh. Ehksp. Teor. Fiz. **35**, 1417–1425 (1958).

[14] A. Z. Dolginov, *Relativistic spherical functions*, Zh. Ehksp. Teor. Fiz. **30**, 746–755 (1956).

[15] A. Z. Dolginov, I. N. Toptygin, *Relativistic spherical functions. II*, Zh. Ehksp. Teor. Fiz. **37**, 1441–1451 (1959).

[16] A. Z. Dolginov, A. N. Moskalev, *Relativistic spherical functions. III*, Zh. Ehksp. Teor. Fiz. **37**, 1697–1707 (1959).

[17] W. Drechsler, *Geometro-stohastically quantized fields with internal spin variables*, J. Math. Phys. **38**, 5531–5558 (1997).

[18] L. D. Eskin, *On the theory of relativistic spherical functions*, Nauchn. dokl. vys. shcoly **2**, 95–97 (1959).

[19] L. D. Eskin, *On the matrix elements of irreducible representations of the Lorentz group*, Izvestia vuzov, Matem. **6**, 179–184 (1961).

[20] J. Fischer, J. Niederle, R. Raczka, *Generalized Spherical Functions for the Noncompact Rotation Groups*, J. Math. Phys. **7**, 816–821 (1966).

[21] V. A. Fock, Zs. f. Phys. **98**, 145 (1935).

[22] I.M. Gel'fand, Z.Ya. Shapiro, *Representations of the rotation group of three–dimensional space and their applications*, Uspekhi Mat. Nauk **7**, 3–117 (1952).

[23] I.M. Gel'fand, A.M. Yaglom, *General relativistic–invariant equations and infinite–dimensional representations of the Lorentz group*, Zh. Ehksp. Teor. Fiz. **18**, 703–733 (1948).

[24] I. M. Gel'fand, M. I. Graev, N. Ya. Vilenkin, Integral geometry and related questions of group representations (Fizmatgiz, Moskow, 1962) [in Russian].

[25] I. M. Gel'fand, R. A. Minlos, Z. Ya. Shapiro, Representations of the Rotation and Lorentz Groups and their Applications (Pergamon Press, Oxford, 1963).

[26] D. M. Gitman, A. L. Shelepin, *Fields on the Poincaré Group: Arbitrary Spin Description and Relativistic Wave Equations*, Int. J. Theor. Phys. **40**(3), 603–684 (2001).

[27] V. Ya. Golodets, *Matrix elements of irreducible unitary and spinor representations of the homogeneous Lorentz group*, Vesti Academii Navuk Belarusskoi SSR **1**, 19–28 (1961).

[28] M. Goto, F. Grosshans, Semisimple Lie Algebras (Marcel Dekker, New York, 1978).

[29] N. X. Hai, *Harmonic analysis on the Poincaré group, I. Generalized matrix elements*, Commun. Math. Phys. **12**, 331–350 (1969).

[30] N. X. Hai, *Harmonic analysis on the Poincaré group, II. The Fourier transform*, Commun. Math. Phys. **22**, 301–320 (1971).

[31] Harish Chandra, *Infinite irreducible representations of the Lorentz group*, Proc. Royal. Soc. London **A189**, 372–401 (1947).

[32] J. Horn, Math. Ann. **105**, 381–407 (1931).

[33] M. Huszar, *Angular Momentum and Unitary Spinor Bases of the Lorentz Group*, Preprint JINR No. E2-5429, Dubna (1970).

[34] M. Huszar, J. Smorodinsky, *Representations of the Lorentz Group on the Two-Dimensional Complex Sphere and Two-Particle States*, Preprint JINR No. E2-5020, Dubna (1970).

[35] M. Huszar, *Spherical functions of the Lorentz group on the hyperboloids*, Acta Phys. Hung. **58**, 175–185 (1985).

[36] M. Huszar, *Addition theorems for the spherical functions of the Lorentz group*, Acta Phys. Hung. **64**, 361–378 (1988).

[37] M. Jacob, G. C. Wick, Ann. of Phys. 7, 404 (1959).

[38] V. I. Kolomytsev, *The reduction of the irreducible unitary representations of the group $SL(2,\mathbb{C})$ restricted to the subgroup $SU(1,1)$. The additional series.*, Teor. Mat. Fiz. **2**, 210–229 (1970).

[39] A. P. Kotel'nikov, In memorial Lobatchevskii **2**, 37–66 (Kazan, 1927).

[40] A. Kihlberg, *Fields on a homogeneous space of the Poincaré group*, Ann. Inst. Henri Poincaré **13**, 57–76 (1970).

[41] G. I. Kuznetsov, Ya. A. Smorodinsky, *Integral representations for relativistic amplitudes and asymptotic theorems*, Yad. Fiz. **6**, 1308–1312 (1967).

[42] G. I. Kuznetsov, M. A. Liberman, A. A. Makarov, Ya. A. Smorodinsky, *Helicity and unitary representations of the Lorentz group*, Yad. Fiz. **10**, 644–656 (1969).

[43] B. M. Levitan, Expansion in eigenfunctions of differential equations of second order (Gostehizdat, Moscow, 1950) [in Russian].

[44] M. A. Liberman, Ya. A. Smorodinsky, M. B. Sheftel, *Unitary representations of the Lorentz group and functions with spin*, Yad. Fiz. **7**, 202–214 (1968).

[45] F. Lurçat, *Quantum field theory and the dynamical role of spin*, Physics **1**, 95 (1964).

[46] M. A. Naimark, Linear Differential Operators (Gostehizdat, Moscow, 1954) [in Russian].

[47] M. A. Naimark, Linear Representations of the Lorentz Group (Pergamon, London, 1964).

[48] P. Pajas, P. Winternitz, *Representations of the Lorentz Group: New Integral Relations between Legendre Functions*, J. Math. Phys. **11**, 1505–1510 (1970).

[49] V. S. Popov, *On the theory of relativistic transformations of the wave functions and the density matrix of particles with spin*, Zh. Ehksp. Teor. Fiz. **37**, 1116–1126 (1959).

[50] G. Rideau, *On the reduction of the regular representation of the Poincaré group*, Commun. Math. Phys. **3**, 218–227 (1966).

[51] M. E. Rose, Multipole Fields (John Wiley & Sons, New York, 1955).

[52] B. A. Rozenfel'd, Non–Euclidean Geometries (Moscow, 1955) [in Russian].

[53] Yu. B. Rumer, Spinorial Analysis (Moscow, 1936) [in Russian].

[54] Yu. B. Rumer, A. I. Fet, Group Theory and Quantized Fields (Moscow, 1977) [in Russian].

[55] L. Ryder, Quantum Field Theory (Cambridge University Press, Cambridge, 1985).

[56] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper & Row, New York, 1961).

[57] A. Sciarrino, M. Toller, J. Math. Phys. **8**, 1252 (1967).

[58] I. S. Shapiro, *Expansion of wave function in irreducible representations of the Lorentz group*, Doklady AN SSSR **106**, 647–649 (1956).

[59] I. S. Shapiro, *Expansion of scattering amplitude in relativistic spherical functions*, Zh. Ehksp. Teor. Fiz. **43**, 1727–1730 (1962).

[60] Ya.A. Smorodinsky, M. Huszar, *Representations of the Lorentz group and the generalization of helicity states*, Teor. Mat. Fiz. **4**, 3, 328–340 (1970).

[61] S. Ström, *On the matrix elements of a unitary representation of the homogeneous Lorentz group*, Arkiv för Fysik **29**, 467–483 (1965).

[62] S. Ström, *A note on the matrix elements of a unitary representation of the homogeneous Lorentz group*, Arkiv för Fysik **33**, 465–469 (1967).

[63] M. Toller, *Free quantum fields on the Poincaré group*, J. Math. Phys. **37**, 2694–2730 (1996).

[64] V. V. Varlamov, *Fundamental Automorphisms of Clifford Algebras and an Extension of Dąbrowski Pin Groups*, Hadronic Journal **22**, 497–535 (1999).

[65] V. V. Varlamov, *Discrete Symmetries and Clifford Algebras*, Int. J. Theor. Phys. **40**, No. 4, 769–805 (2001).

[66] V. V. Varlamov, *Clifford Algebras and Discrete Transformations of Spacetime*, Proceedings of Third Siberian Conference on Mathematical Problems of Spacetime Physics of Complex Systems, Novosibirsk, June 20–22 2000. Publishing House of the Institute of Mathematics, Novosibirsk, pp. 97–135, 2001.

[67] V. V. Varlamov, *Hyperspherical Functions and Linear Representations of the Lorentz Group*, Hadronic Journal **25**, 481–508 (2002).

[68] V. V. Varlamov, *General Solutions of Relativistic Wave Equations*, Int. J. Theor. Phys. **42**, No. 3, 583–633 (2003).

[69] V. V. Varlamov, *Relativistic Wave Equations in the Helicity Basis*, Hadronic Journal **26**, 275–298 (2003).

[70] V. V. Varlamov, *Fields on the Lorentz Group: Helicity Basis and Relativistic Wave Equations*, Invited contribution to "Progress in Quantum Physics Research" (Nova Science Publishers, New York), to appear.

[71] I. A. Verdiev, L. A. Dadashev, *Matrix elements of the Lorentz group unitary representation*, Yad. Fiz. **6**, 1094–1099 (1967).

[72] I. A. Verdiev, *Relativistically invariant expansion of spiral scattering amplitude*, Zh. Ehksp. Teor. Fiz. **55**, 1771–1783 (1968).

[73] I. A. Verdiev, Yad. Fiz. **9**, 1310 (1969).

[74] I. A. Verdiev, G. A. Kerimov, Ya. A. Smorodinsky, *Clebsh-Gordan coefficients of the Lorentz group*, Yad. Fiz. **21**, 1351–1361 (1975).

[75] N. Ya. Vilenkin, Ya. A. Smorodinsky, *Invariant expansions of relativistic amplitudes*, Zh. Ehksp. Teor. Fiz. **46**, 1793–1808 (1964).

[76] N. Ya. Vilenkin, Special Functions and the Theory of Group Representations (AMS, Providence, 1968).

[77] N. Ya. Vilenkin, A. U. Klimyk, Representations of Lie Groups and Special Functions, vols. 1–3. (Dordrecht: Kluwer Acad. Publ., 1991–1993).

[78] B. L. van der Waerden, *Spinoranalyse*, Nachr. d. Ces. d. Wiss. Göttingen, 100–109 (1929).

[79] B. L. van der Waerden, Die Gruppentheoretische Methode in der Quantenmechanik (Springer, Berlin, 1932).

[80] H. Weyl, Classical Groups (Princeton University Press, New York, 1946).

[81] P. Winternitz, I. Fris, *Invariant expansions of relativistic amplitudes and subgroups of the proper Lorentz group*, Yad. Fiz. **1**, 889–901 (1965).